Hence, the given differential equation is exact and the solution is

$$\int (y\cos x + \sin y + y)dx + \int (\sin x + x\cos y + x)dy = c$$

$$y\sin x + x\sin y + xy = c$$ ■

8. *Solve* $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

➡ $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$M = 5x^4 + 3x^2y^2 - 2xy^3$ $\quad N = 2x^3y - 3x^2y^2 - 5y^4$

$\dfrac{\partial M}{\partial y} = 6x^2y - 6xy^2$ $\quad \dfrac{\partial N}{\partial x} = 6x^2y - 6xy^2$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence, the given equation is exact and the solution is

$$\int (5x^4 + 3x^2y^2 - 2xy^3)dx + \int (-5y^4)dy = 0$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = c$$ ■

9. *Solve* $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x\sin y]dy = 0$

➡ $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x\sin y]dy = 0$

$M = y\left(1+\dfrac{1}{x}\right) + \cos y$ $\quad N = x + \log x - x\sin y$

$\dfrac{\partial M}{\partial y} = \left(1+\dfrac{1}{x}\right) - \sin y$ $\quad \dfrac{\partial N}{\partial x} = 1 + \dfrac{1}{x} - \sin y$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence, the given differential equation is exact and the solution is

$$\int \left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + \int [x + \log x - x\sin y]\,dy = c$$

$$yx + y\log x + x\cos y = c$$ ■

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10. *Solve*

$$[\cos x \tan y + \cos(x+y)]\ dx + [\sin x . \sec^2 y + \cos(x+y)]\ dy = 0$$

➡ $[\cos x \tan y + \cos(x+y)]dx + [\sin x . \sec^2 y + \cos(x+y)]\ dy = 0$

$$M = \cos x \tan y + \cos(x+y)$$

$$N = \sin x \sec^2 y + \cos(x+y)$$

$$\frac{\partial M}{\partial y} = \cos x . \sec^2 y - \sin(x+y)$$

$$\frac{\partial N}{\partial x} = \cos x . \sec^2 y - \sin(x+y)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, the given differential equation is exact and the solution is

$$\int [\cos x . \tan y + \cos(x+y)]dx + \int [\sin x \sec^2 y + \cos(x+y)]dy = c$$

$$\sin x . \tan y + \sin(x+y) = c$$ ■

11. *Solve* $\left(6x^2 + 2xy - 2xy\, e^{-x^2}\right)dx + \left(e^{-x^2} + x^2 + 3y^2\right)dy = 0$

➡ $\left(6x^2 + 2xy - 2xy\, e^{-x^2}\right)dx + \left(e^{-x^2} + x^2 + 3y^2\right)dy = 0$

$$M = 6x^2 + 2xy - 2xye^{-x^2} \qquad N = e^{-x^2} + x^2 + 3y^2$$

$$\frac{\partial M}{\partial y} = 2x - 2xe^{-x^2} \qquad \frac{\partial N}{\partial x} = -2xe^{-x^2} + 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, the given differential equation is exact and the solution is

$$\int \left(6x^2 + 2xy - 2xye^{-x^2}\right)dx + \int \left(e^{-x^2} + x^2 + 3y^2\right)dy = c$$

$$2x^3 + x^2 y - y\int 2xe^{-x^2}\, dx + y^3 = c$$

$$2x^3 + x^2 y + ye^{-x^2} + y^3 = c$$ ■

12. *Solve* $\left(x^2 - y^2\right)dx = 2xydy$.

➡ $\left(x^2 - y^2\right)dx = 2xydy$

$$\left(x^2 - y^2\right)dx - 2xydy = 0$$

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$$M = x^2 - y^2 \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = -2y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore, the given equation is exact and the solution is

$$\int \left(x^2 - y^2\right) dx - \int 2xy dy = c$$

$$\frac{x^3}{3} - xy^2 = c$$

Exercises

Solve the following

1) $\left(x^2 - 4xy - 2y^2\right)dx + \left(y^2 - 4xy - 2x^2\right)dy = 0$
2) $\left(x^2 - ay\right)dx + \left(y^2 - ax\right)dy = 0$
3) $(ax + hy + g)dx + (hx + by + f)dy = 0$
4) $\left(x - 2e^y\right)dy + (y + x\sin x)dx = 0$
5) $3x(xy - 2)dx + \left(x^3 + 2y\right)dy = 0$
6) $\left(e^y + 1\right)\cos x dx + e^y \sin x dy = 0$
7) $\left(\frac{2x}{y^3}\right)dx + \left\{\frac{\left(y^2 - 3x^2\right)}{y^4}\right\}dy = 0$
8) $\left(3x^2y^2 + x^2\right)dx + \left(2x^3y + y^2\right)dy = 0$
9) $2xydy + \left(a + y^2\right)dx = 0$
10) $(3 - x + 2xy)dy + \left(3x^2 - y + y^2\right)dx = 0$
11) $\left(x^3 + 2y - y^2\right)dy + 3x^2ydx = 0$
12) $[x\sin xy + \cos(x + y) - \sin y]dy$
    $+ [y\sin xy + \cos(x + y) + \cos x]dx = 0$
13) $\left(e^x + xe^y\right)dy + \left(ye^x + e^y\right)dx = 0$
14) $\left(2 - 6xy + x^2\sec^2 y\right)dy + \left(2x\tan y - 3y^2\right)dx = 0$
15) $xdy + \left(y - x^3\right)dx = 0$

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Answers

1) $x^3 + y^3 - 6xy(x+y) = c$

2) $x^3 + y^3 - 3axy = c$

3) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

4) $xy - x\cos x + \sin x - 2e^y = c$

5) $x^3 y - 3x^2 + y^2 = c$

6) $(e^y + 1)\sin x = c$

7) $x^2 - y^2 = cy^3$

8) $x^3 + y^3 + 3x^3 y^2 = c$

9) $x(a + y^2) = c$

10) $x^3 + xy^2 - xy + 3y = c$

11) $y(3x^3 + 3y - y^2) = c$

12) $\cos xy - \sin(x+y) - \sin x - \cos y = c$

13) $xe^y + ye^x = c$

14) $x^2 \tan y + 2y = c + 3xy^2$

15) $4xy = x^4 + c$

Equations Reducible to Exact Equations

If $Mdx + Ndy = 0$ is not exact i.e., $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, then it can be made exact by multiplying with an integrating factor (I.F). The following methods are followed for finding the I.F.

1. If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of x only, say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor.

2. If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is a function of y only, say $g(y)$, then $e^{\int g(y)dy}$ is an integrating factor.

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3. If an equation is of the form $f_1(x,y)ydx + f_2(x,y)xdy = 0$

   i.e., $M = yf_1, N = xf_2$ and if $Mx - Ny \neq 0$, then $\dfrac{1}{Mx - Ny}$ is an integrating factor.

Worked Examples

1. *Solve* $3x(x+y^2)dy + (x^3 - 3xy - 2y^3)dx = 0$

➡ $3x(x+y^2)dy + (x^3 - 3xy - 2y^3)dx = 0$

i.e., $(x^3 - 3xy - 2y^3)dx + (3x^2 + 3xy^2)dy = 0$

$$M = x^3 - 3xy - 2y^3 \qquad N = 3x^2 + 3xy^2$$

$$\frac{\partial M}{\partial y} = -3x - 6y^2 \qquad \frac{\partial N}{\partial x} = 6x + 3y^2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -9x - 9y^2$$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-9(x+y^2)}{3x(x+y^2)}$$

$$= -\frac{3}{x} = f(x)$$

Therefore, I.F. $= e^{\int f(x)dx} = e^{\int -\frac{3}{x}dx} = e^{\log x^{-3}} = x^{-3} = \dfrac{1}{x^3}$

Multiplying this I.F to the given equation, we get

$$(1 - 3x^{-2}y - 2x^{-3}y^3)dx + \left(3\frac{1}{x} + x^{-2}y^2\right)dy = 0$$

This is an exact differential equation, and the solution of the given equation is

$$\int (1 - 3x^{-2}y - 2x^{-3}y^3)dx + \int \left(\frac{3}{x} + x^{-2}y^2\right)dy = c$$

treating y as constant   the terms do no contain x

$$x - 3\frac{x^{-1}}{-1}y - 2\frac{x^{-2}}{-2}y^3 = c$$

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$$x+\frac{3y}{x}+\frac{y^3}{x^2}=c$$

$$x^3+3xy+y^3=cx^2$$

■

2. *Solve* $[1+(x+y)\tan y]\frac{dy}{dx}+1=0$

➡ $[1+(x+y)\tan y]\frac{dy}{dx}+1=0$

i.e., $[1+(x+y)\tan y]dy+dx=0$

or $dx+[1+(x+y)\tan y]dy=0$

$M=1$ $\qquad N=1+(x+y)\tan y$

$\frac{\partial M}{\partial y}=0$ $\qquad \frac{\partial N}{\partial x}=\tan y$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\tan y=g(y)$$

$$\therefore I.F.=e^{\int g(y)dy}=e^{\int \tan y dy}=e^{\log(\sec y)}$$
$$=\sec y$$

Multiplying the I.F to the given equation we get.

$$\sec y dx+[\sec y+(x+y)\sec y\tan y]dy=0$$

This is an exact equation and the solution of the given equation is

$$\int \sec y dx + \int(\sec y+x\sec y+y\sec y\tan y)dy=c$$

treating y as a constant $\qquad$ the terms do not contain x

$$x\sec y+\log(\sec y+\tan y)+\int y(\sec y\tan y)\,dy=c$$

$$x\sec y+\log(\sec y+\tan y)+y\sec y-\int 1.\sec y dy=c$$

$$x\sec y+\log(\sec y+\tan y)+y\sec y-\log(\sec y+\tan y)=c$$

$$x\sec y+y\sec y=c$$

$$(x+y)\sec y=c$$

■

3. *Solve* $(1+xy)ydx+(1-xy)xdy=0$

➡ $(1+xy)ydx+(1-xy)xdy=0$

$$M=(1+xy)y=yf_1(x,y)$$

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$$N = (1 - xy)x = xf_2(x, y)$$

$$\therefore Mx - Ny = xy(1 + xy) - xy(1 - xy)$$

$$= xy + x^2y^2 - xy + x^2y^2$$

$$= 2x^2y^2 \neq 0$$

$$\therefore I.F = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2} \text{ or } \frac{1}{x^2y^2}$$

Multiplying I.F to the given equation, we get

$$\frac{(1 + xy)y}{x^2y^2}dx + \frac{(1 - xy)x}{x^2y^2}dy = 0$$

$$\left(\frac{x^{-2}}{y} + \frac{1}{x}\right)dx + \left(x^{-1}y^{-2} - \frac{1}{y}\right)dy = 0$$

This is an exact equation and the solution of the given equation is,

$$\int\left(\frac{1}{y}x^{-2} + \frac{1}{x}\right)dx + \int\left(x^{-1}y^{-2} - \frac{1}{y}\right)dy = c$$

treating y as constant   the terms do not contain x

$$\frac{1}{y}\frac{x^{-1}}{-1} + \log x - \log y = c$$

$$-\frac{1}{xy} + \log x - \log y = c$$ ■

Note 1. $xdy + ydx = d(xy)$

2. $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

3. $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

4. $g(y)f'(x)dx + g'(y)f(x)dy = d[f(x)g(y)]$

4. *Solve* $xdy - ydx + a(x^2 + y^2)dx = 0$

➡ $xdy - ydx + a(x^2 + y^2)dx = 0$

$$\Rightarrow \frac{xdy - ydx}{x^2} + \frac{a(x^2 + y^2)}{x^2}dx = 0$$

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i.e., $d\left(\frac{y}{x}\right)+a\left[1+\left(\frac{y}{x}\right)^2\right]dx=0$

Put $\frac{y}{x}=v$

$$dv+a\left(1+v^2\right)dx=0$$

Which is variable separable form,

$$dv=-a\left(1+v^2\right)dx$$

$$\frac{dv}{1+v^2}=-a\,dx$$

$$\int\frac{dv}{1+v^2}=-a\int dx+c$$

$$\tan^{-1}v=-ax+c$$

$$\tan^{-1}\left(\frac{y}{x}\right)+ax=c$$

■

5. *Solve* $\left(x^4+y^2\right)dy=4x^3ydx$

➨ $\left(x^4+y^2\right)dy=4x^3ydx$

$$x^4dy+y^2dy=4x^3ydx$$

$$y^2dy=4x^3ydx-x^4dy$$

$$dy=\frac{y\left(4x^3dx\right)-x^4dy}{y^2}$$

$$dy=d\left(\frac{x^4}{y}\right)$$

$$\int dy=\int d\left(\frac{x^4}{y}\right)+c$$

$$y=\frac{x^4}{y}+c$$

■

6. *Solve* $ydx-xdy-3x^2y^2e^{x^3}dx=0$

➨ $ydx-xdy-3x^2y^2e^{x^3}dx=0$

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Dividing by y^2 on both sides, we get

$$\frac{ydx - xdy}{y^2} - 3x^2 e^{x^3} dx = 0$$

$$d\left(\frac{x}{y}\right) - 3x^2 e^{x^3} dx = 0$$

$$\int d\left(\frac{x}{y}\right) - \int 3x^2 e^{x^3} dx = c$$

$$\frac{x}{y} - \int e^t dt = c \qquad x^3 = t \Rightarrow 3x^2 dx = dt$$

$$\frac{x}{y} - e^t = c$$

$$\frac{x}{y} - e^{x^3} = c$$

■

7. *Solve* $(x+y)^2\left[x\frac{dy}{dx} + y\right] = xy\left[1 + \frac{dy}{dx}\right]$

➡ $(x+y)^2\left[x\frac{dy}{dx} + y\right] = xy\left[1 + \frac{dy}{dx}\right]$ ---(1)

Put $x + y = t$ and $xy = z$

$$1 + \frac{dy}{dx} = \frac{dt}{dx} \qquad y + x\frac{dy}{dx} = \frac{dz}{dx}$$

(1) becomes, $t^2\frac{dz}{dx} = z\frac{dt}{dx}$

$$\frac{dt}{t^2} = \frac{dz}{z}$$

Integrating the above equation, we get

$$\int \frac{dt}{t^2} = \int \frac{dz}{z}$$

$$-\frac{1}{t} = \log z + c \Rightarrow -\frac{1}{(x+y)} = \log(xy) + c$$

■

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