$$\therefore J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix}$$

$$J = x - y$$

From the given equations, we have

$$x = \frac{v}{y} \qquad \therefore u = \frac{v}{y} + y$$

$$\Rightarrow uy = v + y^2$$

$$\therefore y^2 - uy + v = 0$$

$$\Rightarrow y = \frac{u \pm \sqrt{u^2 - 4v}}{2}$$

Considering any one of the roots of y, we get

$$y = \frac{u + \sqrt{u^2 - 4v}}{2}$$

Substituting this in $u = x + y \Rightarrow x = u - y$, we get

$$= u - \frac{u + \sqrt{u^2 - 4v}}{2}$$

$$x = \frac{2u - u - \sqrt{u^2 - 4v}}{2}$$

$$\therefore x = \frac{u - \sqrt{u^2 - 4v}}{2}$$

$$\therefore \frac{\partial x}{\partial u} = \frac{1}{2}\left[1 - \frac{1}{2\sqrt{u^2 - 4v}}(2u)\right] = \frac{\sqrt{u^2 - 4v} - u}{2\sqrt{u^2 - 4v}} = \frac{-x}{\sqrt{u^2 - 4v}}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}\left[-\frac{1}{2\sqrt{u^2 - 4v}}(-4)\right] = \frac{1}{\sqrt{u^2 - 4v}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\left[1 + \frac{1}{2\sqrt{u^2 - 4v}}(2u)\right] = \frac{1}{2}\left[\frac{\sqrt{u^2 - 4v} + u}{\sqrt{u^2 - 4v}}\right] = \frac{y}{\sqrt{u^2 - 4v}}$$

$$\frac{\partial y}{\partial v}=\frac{1}{2}\left[\frac{1}{2\sqrt{u^2-4v}}(-4)\right]=-\frac{1}{\sqrt{u^2-4v}}$$

$$\therefore J'=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{-x}{\sqrt{u^2-4v}} & \frac{1}{\sqrt{u^2-4v}}\\ \frac{y}{\sqrt{u^2-4v}} & -\frac{1}{\sqrt{u^2-4v}}\end{vmatrix}$$

$$=\frac{x}{u^2-4v}-\frac{y}{u^2-4v}=\frac{x-y}{(x-y)^2} \qquad \because x-y=\sqrt{u^2-4v}$$

$$\therefore J'=\frac{1}{(x-y)}$$

$$\therefore JJ'=(x-y)\times\frac{1}{(x-y)}=1$$

12. *If $x=\frac{u^2}{v}$, $y=\frac{v^2}{u}$. Find the associated increase Jacobian*

➡ $x=\frac{u^2}{v}$, $\quad y=\frac{v^2}{u}$

$\Rightarrow u^2=xv$, $\quad v^2=yu$ ---(1)

$v^4=y^2u^2$

Substituting for u^2, we get $v^4=y^2xv$

$v^3=xy^2$

$\Rightarrow v=y^{\frac{2}{3}}x^{\frac{1}{3}}$

$$u^2=xv=x\left(y^{\frac{2}{3}}x^{\frac{1}{3}}\right)=y^{\frac{2}{3}}x^{\frac{4}{3}}$$

$$u=y^{\frac{1}{3}}x^{\frac{2}{3}} \quad\Rightarrow \frac{\partial u}{\partial x}=\frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}}, \quad \frac{\partial u}{\partial y}=\frac{1}{3}\cdot x^{\frac{2}{3}}\cdot y^{-\frac{2}{3}}$$

$$v=y^{\frac{2}{3}}x^{\frac{1}{3}} \quad\Rightarrow \frac{\partial v}{\partial x}=\frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}, \quad \frac{\partial v}{\partial y}=\frac{2}{3}\cdot y^{-\frac{1}{3}}\cdot x^{\frac{1}{3}}$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2}{3}x^{-\frac{1}{3}} \cdot y^{\frac{1}{3}} & \frac{1}{3}x^{\frac{2}{3}} \cdot y^{-\frac{2}{3}} \\ \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}} & \frac{2}{3}y^{-\frac{1}{3}}x^{\frac{1}{3}} \end{vmatrix}$$

$$= \frac{4}{9}x^{-\frac{1}{3}}y^{\frac{1}{3}} \cdot x^{\frac{1}{3}}y^{-\frac{1}{3}} - \frac{1}{9}x^{\frac{2}{3}}y^{-\frac{2}{3}} \cdot x^{-\frac{2}{3}}y^{\frac{2}{3}}$$

$$= \frac{4}{9}(1) - \frac{1}{9}(1) = \frac{1}{3}$$ ■

13. *If* $x = r\cos\theta$, $y = r\sin\theta$, *find* $\frac{\partial(r,\theta)}{\partial(x,y)}$ *in terms of r*

➡ $x = r\cos\theta$ $\qquad y = r\sin\theta$

$$\Rightarrow r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}.2x = \frac{x}{\sqrt{x^2+y^2}}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{1}{(x^2+y^2)/x^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{1/x}{(x^2+y^2)/x^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{1}{(x^2+y^2)^{\frac{1}{2}}} = \frac{1}{r}$$ ■

14. *If* $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ *and* $x = r\cos\theta$, $y = r\sin\theta$ *show that* $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^2 \sin 2\theta$

➡ $u = x^2 - 2y^2 \qquad v = 2x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 4x$$

$$\frac{\partial u}{\partial y} = -4y \qquad \frac{\partial v}{\partial y} = -2y$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} = -4xy + 16xy$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 12xy$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

We have, $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)}$

$$= (12xy)(r) = 12r(r\cos\theta)(r\sin\theta)$$

$$= 12r^3(\sin\theta.\cos\theta)$$

$$= 12r^3\left(\frac{\sin 2\theta}{2}\right)$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$$ ■

15. *If* $u = x^2 - y^2$, $v = 2xy$ *and* $x = r\cos\theta$, $y = r\sin\theta$, *then determine the Jacobian* $\frac{\partial(u,v)}{\partial(r,\theta)}$

➡ $$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

We have, $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = (4x^2 + 4y^2)(r) = 4(x^2 + y^2).r$$

$$= 4(r^2)\cdot r = 4r^3 \qquad \because x^2 + y^2 = r^2$$ ■

16. *If* $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ *find* $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

➡ $$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial z}{\partial r} = 0$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \qquad \frac{\partial z}{\partial \theta} = 0$$

$$\frac{\partial x}{\partial z} = 0 \qquad \frac{\partial y}{\partial z} = 0 \qquad \frac{\partial z}{\partial z} = 1$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta(r\cos\theta - 0) + r\sin\theta(\sin\theta - 0) + 0$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

■

17. *If* $u = x - y$ *and* $v = \dfrac{1}{x-y}$, *show that* $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

$$u = x - y \qquad v = \frac{1}{x-y}$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = -\frac{1}{(x-y)^2}$$

$$\frac{\partial u}{\partial y} = -1 \qquad \frac{\partial v}{\partial y} = -\frac{1(-1)}{(x-y)^2} = \frac{1}{(x-y)^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ \frac{-1}{(x-y)^2} & \frac{1}{(x-y)^2} \end{vmatrix}$$

$$= \frac{1}{(x-y)^2} - \frac{1}{(x-y)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

■

18. *If* $u = \tan^{-1} x + \tan^{-1} y,\ \ v = \dfrac{x+y}{1-xy}$ *show that* $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

➡ $u = \tan^{-1} x + \tan^{-1} y$

$$\frac{\partial u}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial u}{\partial y} = \frac{1}{1+y^2}$$

$$v = \frac{x+y}{1-xy}$$

$$\frac{\partial v}{\partial x} = \frac{(1-xy)1-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(1-xy)1-(x+y)(-x)}{(1-xy)^2} = \frac{1-xy+x^2+xy}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{1+x^2}\,\frac{1+x^2}{(1-xy)^2} - \frac{1}{1+y^2}\,\frac{1+y^2}{(1-xy)^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$ ■

19. *If* $u = x + 3y^2 - z^3,\ v = 4x^2yz,\ w = 2z^2 - xy$, *prove that* $\dfrac{\partial(u,v,w)}{\partial(x,y,z,)}$ *at* $(1,-1,0)$ *is* 20

➡
$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 8xyz \qquad \frac{\partial w}{\partial x} = -y$$

$$\frac{\partial u}{\partial y} = 6y \qquad \frac{\partial v}{\partial y} = 4x^2 z \qquad \frac{\partial w}{\partial y} = -x$$

$$\frac{\partial u}{\partial z} = -3z^2 \qquad \frac{\partial v}{\partial z} = 4x^2 y \qquad \frac{\partial w}{\partial z} = 4z$$

We have, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ at } (1,-1,0) = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1(0-4) + 6(0+4) + 0$$

$$= -4 + 24 = 20$$ ■

Exercises

1) *Find* $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ *for the following*

(a) $u = x^2 - 2y,\ v = x + y + z,\ w = x - 2y + 3z$

(b) $u = xyz,\ v = xy + yz + zx,\ w = x + y + z$

(c) $u = \cos x,\ v = \sin x \cos y,\ w = \sin x \sin y \cos z$

(d) $u = xy^2,\ v = yz^2, w = zx^2$

(e) $u = x \cos y \cos z,\ v = x \cos y \sin z,\ w = x \sin y$

2) *If* $u = \sqrt{x^2 + y^2},\ v = \sqrt{x^2 - y^2}$ *then prove that* $JJ' = 1$

3) *If* $u = x + y,\ v = xy$ *find* $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ & $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$ *and verify* $JJ' = 1$

4) *If* $u = \dfrac{xy}{z},\ v = \dfrac{yz}{x},\ w = \dfrac{zx}{y}$ *find* $J = \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ *verify* $JJ' = 1$

5) *Find* $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ *of*

(a) $x = \dfrac{1}{2}(u^2 - v^2),\ y = uv,\ z = w$

(b) $x = u,\ y = u\tan v,\ z = w$

6) *If* $u = 2axy$ & $v = a\left(x^2 - y^2\right)$ *where* $x = r\cos\theta,\ y = r\sin\theta$ *then, prove that* $\dfrac{\partial(u,v)}{\partial(r,\theta)} = -4a^2r^3$

7) *If* $u = x^2 - y^2$ & $v = 2xy$ *where* $x = r\cos\theta,\ y = r\sin\theta$, *show that* $\dfrac{\partial(u,v)}{\partial(r,\theta)}$

8) *If* $u = a(x+y),\ v = b(x-y)$ & $x = r^2\cos 2\theta,\ y = r^2\sin 2\theta$ *then, find* $\dfrac{\partial(u,v)}{\partial(r,\theta)}$

9) *If* $x = \dfrac{u^2 v}{v},\ y = \dfrac{v^2}{u}$ *find* $\dfrac{\partial(u,v)}{\partial(x,y)}$

10) *If* $u = x + y,\ v = \dfrac{1}{x+y}$ *then,* $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

11) *If* $x = u^2,\ y = v^2 - w^2,\ z = w^2 - u^2$, *show that* $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 0$

12) *If* $ux = 2yz,\ vy = 2zx,\ wz = 2xy$, *prove that* $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

13) *If* $u = \sin^{-1}x + \sin^{-1}y,\quad v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ *show that* $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

Answers

1) (a) $10x + 4$ (b) $(x-y)(y-z)(z-x)$

(c) $-\sin^3 x\sin^2 y\sin z$ (d) $9x^2y^2z^2$

(e) $-x^2 y$

8) $-8abr^3$

9) $\dfrac{1}{3}$