7. *Prove that* $\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$

➡ We know that $\beta\left(n+\frac{1}{2},n+\frac{1}{2}\right)=2\int_0^{\pi/2}\sin^{2n}\theta\cos^{2n}\theta\,d\theta$

$$=2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2n}d\theta$$

$$=2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2n}d\theta$$

$$=\frac{1}{2^{2n-1}}\int_0^{\pi/2}\sin^{2n}2\theta\,d\,\theta$$

Put $2\theta=x\Rightarrow 2d\theta=dx\Rightarrow d\theta=dx/2$

If $\theta=0\Rightarrow x=0$, if $\theta=\pi/2\Rightarrow x=\pi$

$$\beta\left(n+\frac{1}{2},n+\frac{1}{2}\right)=\frac{1}{2^{2n-1}}\int_0^{\pi}\sin^{2n}x\frac{dx}{2}$$

$$=\frac{1}{2^{2n-1}}\frac{1}{2}\cdot 2\int_0^{\pi/2}\sin^{2n}xdx$$

$$=\frac{1}{2^{2n-1}}\frac{1}{2}\beta\left(\frac{2n+1}{2},\frac{1}{2}\right)\quad\left[\because 2\int_0^{\pi/2}\sin^{p}\theta=\beta\left(\frac{p+1}{2},\frac{1}{2}\right)\right]$$

$$\beta\left(n+\frac{1}{2},n+\frac{1}{2}\right)=\frac{1}{2^{2n}}\beta\left(n+\frac{1}{2},\frac{1}{2}\right)$$

$$\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+n+\frac{1}{2}\right)}=\frac{1}{2^{2n}}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2n+\frac{1}{2}+\frac{1}{2}\right)}$$

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)}=\frac{1}{2^{2n}}\frac{\sqrt{\pi}}{\Gamma(n+1)}$$

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{2n\Gamma(2n)}=\frac{\sqrt{\pi}}{2^{2n}\ n\Gamma(n)}$$

$$\therefore \Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$$ ■

8. *Prove that* $\beta\left(m,\frac{1}{2}\right)=2^{2m-1}\beta(m,m)$

➡ We have $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$ ---(1)

Substituting $n=\frac{1}{2}$ in (1), we get

$$\beta\left(m,\frac{1}{2}\right)=2\int_0^{\pi/2}\sin^{2m-1}\theta\, d\theta \quad \text{---(2)}$$

Substituting $n=m$ in (1), we get

$$\beta(m,m)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2m-1}\theta d\theta=2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2m-1}d\theta$$

$$=2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2m-1}d\theta=\frac{2}{2^{2m-1}}\int_0^{\pi/2}\sin^{2m-1}2\theta d\theta$$

Put $2\theta=x\Rightarrow d\theta=dx/2$

If $\theta=0\Rightarrow x=0$, if $\theta=\pi/2\Rightarrow x=\pi$

$$\beta(m,m)=\frac{2}{2^{2m-1}}\int_0^{\pi}\sin^{2m-1}x\frac{dx}{2}$$

$$=\frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2}\sin^{2m-1}x dx=\frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2}\sin^{2m-1}\theta d\theta$$

$$\beta(m,m)=\frac{1}{2^{2m-1}}\beta\left(m,\frac{1}{2}\right)$$

$$\beta\left(m,\frac{1}{2}\right)=2^{2m-1}\beta(m,m)$$ ■

9. *Prove that* $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$

➡ Let $I_1 = \int_0^\infty \sqrt{x} e^{-x^2} dx = \int_0^\infty x^{\frac{1}{2}} e^{-x^2} dx$

Put $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{1}{2\sqrt{t}} dt$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore I_1 = \int_0^\infty \left(\sqrt{t}\right)^{\frac{1}{2}} e^{-t} \frac{1}{2\sqrt{t}} dt = \frac{1}{2}\int_0^\infty \left(\sqrt{t}\right)^{\frac{1}{2}-1} e^{-t} dt$$

$$= \frac{1}{2}\int_0^\infty \left(\sqrt{t}\right)^{-\frac{1}{2}} e^{-t} dt = \frac{1}{2}\int_0^\infty t^{\frac{-1}{4}} e^{-t} dt$$

$$= \frac{1}{2}\Gamma\left(-\frac{1}{4}+1\right) \qquad \left[\because \int_0^\infty x^n e^{-x} dx = \Gamma(n+1)\right]$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

Let $I_2 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$

Put $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{1}{2\sqrt{t}} dt$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$I_2 = \int_0^\infty \frac{e^{-t}}{\left(\sqrt{t}\right)^{\frac{1}{2}}} \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t}}{\left(\sqrt{t}\right)^{\frac{3}{2}}} dt = \frac{1}{2}\int_0^\infty t^{\frac{-3}{4}} e^{-t} dt = \frac{1}{2}\Gamma\left(-\frac{3}{4}+1\right)$$

$$I_2 = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

Therefore,

$$\int_0^\infty \sqrt{xe^{-x^2}}\,dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$$

$$= I_1 \times I_2 = \frac{1}{2}\Gamma\left(\frac{3}{4}\right) \times \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} \qquad \left[\because \Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{4\left(\frac{1}{\sqrt{2}}\right)} = \frac{\pi}{2\sqrt{2}}$$ ■

10. *Prove that* $\int_0^\infty x^2 e^{-x^4}\,dx \times \int_0^\infty e^{-x^4}\,dx = \frac{\pi}{8\sqrt{2}}$

➡ Let $I_1 = \int_0^\infty x^2 e^{-x^4}\,dx$

Put $x^4 = t \Rightarrow x = t^{1/4} \Rightarrow dx = \frac{1}{4}t^{-(3/4)}dt$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore I_1 = \int_0^\infty t^{\frac{1}{2}} e^{-t} \frac{1}{4} t^{-\frac{3}{4}}\,dt$$

$$= \frac{1}{4}\int_0^\infty t^{-\frac{1}{4}} e^{-t}\,dt = \frac{1}{4}\Gamma\left(-\frac{1}{4}+1\right)$$

$$I_1 = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

Let $I_2 = \int_0^\infty e^{-x^4}\,dx$

Put $x^4 = t \Rightarrow x = t^{\frac{1}{4}} \Rightarrow dx = \frac{1}{4} t^{-\frac{3}{4}} dt$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore I_2 = \int_0^\infty e^{-t} \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$= \frac{1}{4} \int_0^\infty t^{-\frac{3}{4}} e^{-t} dt = \frac{1}{4} \Gamma\left(-\frac{3}{4} + 1\right)$$

$$I_2 = \frac{1}{4} \Gamma\left(\frac{1}{4}\right)$$

Therefore,

$$\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx = I_1 \times I_2 = \left[\frac{1}{4} \Gamma\left(\frac{3}{4}\right)\right] \left[\frac{1}{4} \Gamma\left(\frac{1}{4}\right)\right]$$

$$= \frac{1}{16} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) = \frac{1}{16} \Gamma\left(1 - \frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{16} \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)}$$

$$= \frac{1}{16} \frac{\pi}{(1/\sqrt{2})} = \frac{\pi}{8\sqrt{2}}$$ ■

11. *Evaluate the following integrals*

(i) $\int_0^{\pi/2} \cos^7 \theta \, d\theta$ (ii) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos \theta}}$

(iii) $\int_0^{\pi/2} \sqrt{\cot \theta} \, d\theta$ (iv) $\int_0^{\pi/2} \sqrt{\tan \theta} \, d\theta$

(v) $\int_0^2 x(8 - x^3)^{\frac{1}{3}} dx$

Note (1) $$\int_0^{\pi/2} \sin^p \theta \cdot \cos^q \theta d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

(2) $$\int_0^{\pi/2} \sin^p \theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1}{2}\right)$$

(3) $$\int_0^{\pi/2} \cos^p \theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1}{2}\right)\Bigg]$$

➡

(i) $$\int_0^{\pi/2} \cos^7 \theta \, d\theta = \frac{1}{2}\beta\left(\frac{7+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\beta\left(4, \frac{1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)} = \frac{1}{2}\frac{3!\sqrt{\pi}}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{3\sqrt{\pi}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{48\sqrt{\pi}}{105\sqrt{\pi}} = \frac{16}{35}$$

(ii) $$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}} = \int_0^{\pi/2} \cos^{-\frac{1}{2}} \theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{-(1/2)+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

(iii) $$\int_0^{\pi/2}\sqrt{\cot\ \theta}\,d\theta=\int_0^{\pi/2}\sqrt{\frac{\cos\theta}{\sin\theta}}\,d\theta$$

$$=\int_0^{\pi/2}\cos^{\frac{1}{2}}\theta\cdot\sin^{-\frac{1}{2}}\theta\,d\,\theta$$

$$=\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2},\frac{\frac{-1}{2}+1}{2}\right)$$

$$=\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)=\frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$=\frac{1}{2}\frac{\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}=\frac{1}{2}\frac{\pi}{\sin\left(\frac{1}{4}\pi\right)}$$

$$=\frac{1}{2}\frac{\pi}{\frac{1}{\sqrt{2}}}=\frac{\pi}{\sqrt{2}}$$

(iv) $$\int_0^{\pi/2}\sqrt{\tan\theta}\ d\theta=\int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\,\theta$$

$$=\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2},\frac{-\frac{1}{2}+1}{2}\right)$$

$$=\frac{1}{2}\ \beta\left(\frac{3}{4},\frac{1}{4}\right)$$ See the previous example

$$=\frac{\pi}{\sqrt{2}}$$

(v) Let $I=\int_0^2 x\left(8-x^3\right)^{\frac{1}{3}}dx$

Put $x^3=8t \Rightarrow x=2t^{\frac{1}{3}} \Rightarrow dx=2\frac{1}{3}t^{-\frac{2}{3}}dt$

If $x=0\Rightarrow t=0$, if $x=2\Rightarrow t=1$

$$\therefore I=\int_0^1 2t^{\frac{1}{3}}(8-8t)^{\frac{1}{3}}\frac{2}{3}t^{-\frac{2}{3}}dt$$

$$=\frac{4}{3}\int_0^1 t^{-\frac{1}{3}}8^{\frac{1}{3}}(1-t)^{\frac{1}{3}}dt=\frac{8}{3}\int_0^1 t^{-\frac{1}{3}}(1-t)^{\frac{1}{3}}dt$$

$$=\frac{8}{3}\beta\left(-\frac{1}{3}+1,\frac{1}{3}+1\right)=\frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right)$$

$$=\frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}=\frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}+1\right)}{\Gamma(2)}$$

$$=\frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{1}=\frac{8}{9}\Gamma\left(1-\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)=\frac{8}{9}\frac{\pi}{\sin(\pi/3)}$$

$$=\frac{8}{9}\frac{\pi}{(\sqrt{3}/2)}=\frac{16\pi}{9\sqrt{3}}$$ ■

12. *Prove that* $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{1}{\sqrt{\sin\theta}}d\theta=\pi$

➡ $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}=\int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\,d\theta\times\int_0^{\pi/2}\sin^{-\frac{1}{2}}\theta\,d\theta$

$$=\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2},\frac{1}{2}\right)\times\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2},\frac{1}{2}\right)$$

$$= \frac{1}{4}\,\beta\left(\frac{3}{4},\frac{1}{2}\right)\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\pi}{\Gamma\left(1+\frac{1}{4}\right)} = \frac{\pi}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi$$

Exercises

1) *Prove the following*

(i) $\beta(4,3) = \frac{1}{60}$ (ii) $\beta\left(\frac{3}{2},\frac{5}{2}\right) = \frac{\pi}{16}$

(iii) $\beta\left(\frac{7}{2},\frac{1}{2}\right) = \frac{5\pi}{16}$ (iv) $\beta\left(\frac{1}{4},\frac{1}{2}\right) = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\sqrt{2}\,\pi}$

(v) $\beta\left(\frac{5}{2},\frac{7}{2}\right) = \frac{3\pi}{256}$ (vi) $\beta\left(\frac{5}{6},\frac{1}{6}\right) = 2\pi$

2) *Evaluate the following*

(i) $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{1}{2}}\,dx$ (ii) $\int_0^1 \sqrt{\frac{x}{1-x}}\,dx$

(iii) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$ (iv) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(v) $\int_0^a x^4 \sqrt{a^2 - x^2}\, dx$ (vi) $\int_0^1 \sqrt{1 - x^4}\, dx$

(vii) $\int_0^1 x^3 \left(1 - \sqrt{x}\right) dx$ (viii) $\int_0^2 \left(4 - x^2\right)^{\frac{3}{2}} dx$

(ix) $\int_0^2 \left(8 - x^3\right)^{-\frac{1}{3}} dx$ (x) $\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}}$

(xi) $\int_0^\infty \frac{x^4}{1 + x^4}\, dx$ (xii) $\int_0^\infty \frac{1}{1 + x^4}\, dx$

(xiii) $\int_{-1}^1 \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}} dx$ (xiv) $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$

(xv) $\int_0^1 \frac{1}{\sqrt{1 - x^4}}\, dx$ (xvi) $\int_0^1 x^5 \left(1 - x^3\right)^3 dx$

3) *Prove the following*

(i) $\int_0^\infty x e^{-x^8} dx \times \int_0^\infty x e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$

(ii) $\int_0^{\pi/2} \sin^p \theta \, d\theta \times \int_0^{\pi/2} \sin^{p+1} d\theta = \frac{\pi}{2(p+1)}$

(iii) $\int_0^{\pi/a} (a - x)^{m-1} x^{n-1} dx = a^{m+n-1} \beta(m,n)$

(iv) $\int_0^\infty \frac{x^3\left(1 - x^5\right)}{(1 + x)^{13}} = 0$

(v) $\int_0^\infty \frac{x^8\left(1 - x^6\right)}{(1 + x)^{24}}\, dx = 0$

(vi) $\int_0^a \frac{dx}{\left(a^n - x^n\right)^{\frac{1}{n}}} = \frac{1}{n} \beta\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$

(vii) $\int_0^\infty \frac{x^{n-1}}{(x+a)^{m+n}}dx = \frac{1}{a^n}\beta(m,n)$

[Hint put $x = at$]

(viii) $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

[Hint put $x^n = t$]

(ix) $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}dx = \frac{\pi}{4\sqrt{2}}$

(x) $\int_0^1 x^m\left(1-x^2\right)^n dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, n+1\right)$

[Hint put $x^2 = t$]

Answers

2) (i) $\frac{\pi}{16}$ (ii) $\frac{\pi}{2}$ (iii) $\frac{64\sqrt{2}}{15}$

(iv) 12π (v) $\frac{\pi a^6}{32}$ (vi) $\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{6\Gamma\left(\frac{3}{4}\right)}$

(vii) $\frac{1}{21}$ (viii) 3π (ix) $\frac{2\pi}{3\sqrt{3}}$

(x) π (xi) $\frac{\pi}{2\sqrt{2}}$ (xii) $\frac{\pi}{2\sqrt{2}}$

(xiii) π (xiv) $\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4a\sqrt{2\pi}}$ (xv) $\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{4\,\Gamma\left(\frac{3}{4}\right)}$

(xvi) $\frac{1}{60}$