$$\frac{e^x}{y} = -\frac{x^3}{3} + c$$

$$\frac{e^x}{y} + \frac{x^3}{3} = c$$

■

6. *Solve* $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$

➡ $$\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$$

$$\frac{dy}{dx} = \frac{x^2 + 1}{2xy} + \frac{y}{2x}$$

i.e., $\frac{dy}{dx} - \frac{y}{2x} = \frac{x^2 + 1}{2xy}$

i.e., $\frac{dy}{dx} - \frac{y}{2x} = \frac{x^2 + 1}{2x} y^{-1}$, dividing by y^{-1} on both sides, we get

$$y\frac{dy}{dx} - \frac{y^2}{2x} = \frac{x^2 + 1}{2x} \quad \text{---(1)}$$

Put $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

$$y\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$$

$\therefore$ (1) becomes, $\frac{1}{2}\frac{dt}{dx} - \frac{t}{2x} = \frac{x^2 + 1}{2x}$

i.e., $\frac{dt}{dx} - \frac{t}{x} = \frac{x^2 + 1}{x}$, this is linear in t,

Here $P = -\frac{1}{x}$, $Q = \frac{x^2 + 1}{x}$

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution of the given equation is

$$t(I.F.) = \int Q(I.F.)dx + c$$

$$y^2\left(\frac{1}{x}\right) = \int \frac{x^2+1}{x}\left(\frac{1}{x}\right)dx + c$$

$$\frac{y^2}{x} = \int \frac{x^2+1}{x^2}dx + c$$

$$\frac{y^2}{x} = \int \left(1 + x^{-2}\right)dx + c = x + \frac{x^{-1}}{-1} + c$$

$$\frac{y^2}{x} = x - \frac{1}{x} + c$$

$$\therefore y^2 = x^2 - 1 + cx$$ ■

7. *Solve* $(y\log x - 2)ydx = xdy$

➡ $(y\log x - 2)ydx = xdy$

$$y^2\log x - 2y = x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2\log x - 2y}{x} = \frac{y^2\log x}{x} - \frac{2y}{x}$$

$\frac{dy}{dx} + \frac{2}{x}y = \frac{\log x}{x}y^2$, dividing by y^2 on both sides, we get

$$y^{-2}\frac{dy}{dx} + \frac{2}{x}y^{-1} = \frac{\log x}{x}$$

Put $y^{-1} = t \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow y^{-2}\frac{dy}{dx} = -\frac{dt}{dx}$$

$$\therefore -\frac{dt}{dx} + \frac{2}{x}t = \frac{\log x}{x}$$

or $\frac{dt}{dx} - \frac{2}{x}t = -\frac{\log x}{x}$, this is linear in *t*.

Here $P = \frac{-2}{x}$, $Q = \frac{-\log x}{x}$

$$\text{I.F.} = e^{\int p\,dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$$

Therefore, solution of the given equation is

$$t(I.F.) = \int Q(I.F.)\,dx + c$$

$$y^{-1}x^{-2} = \int\left(\frac{-\log x}{x}\right)(x^{-2})\,dx + c$$

$$\frac{1}{x^2 y} = -\int \frac{\log x}{x^3}dx + c$$

$$= -\int \log x.\frac{1}{x^3}dx + c$$

$$= -\left[\log x.\frac{x^{-2}}{-2} - \int \frac{1}{x}\,\frac{x^{-2}}{-2}dx\right] + c$$

$$= \frac{\log x}{2x^2} - \frac{1}{2}\int x^{-3}dx + c$$

$$= \frac{\log x}{2x^2} - \frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + c$$

$$\frac{1}{x^2 y} = \frac{\log x}{2x^2} + \frac{1}{4x^2} + c$$

$$\text{or } 4 = 2y\log x + y + c4x^2 y$$ ■

8. *Solve* $\frac{dy}{dx} - y\tan x = y^2 \sec x$

➡ $\frac{dy}{dx} - y\tan x = y^2 \sec x$, dividing y^2 on both sides, we get

$$y^{-2}\frac{dy}{dx} - y^{-1}\tan x = \sec x$$

Put $y^{-1} = t \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow y^{-2}\frac{dy}{dx} = -\frac{dt}{dx}$$

$$-\frac{dt}{dx} - t\tan x = \sec x$$

$$\frac{dt}{dx}+t\tan x=-\sec x$$, this is linear in t.

Here $P=\tan x$, $Q=-\sec x$

$$\text{I.F.}=e^{\int Pdx}=e^{\int \tan x dx}=e^{\log(\sec x)}=\sec x$$

Therefore, the solution of the given equation is

$$t(I.F)=\int Q(I.F)dx+c$$

$$y^{-1}(\sec x)=\int(-\sec x)\sec x dx+c$$

$$\frac{\sec x}{y}=-\tan x+c$$

9. *Solve* $\frac{dz}{dx}+\frac{z}{x}(\log z)=\frac{z}{x^2}(\log z)^2$

➡ $\frac{dz}{dx}+\frac{z}{x}(\log z)=\frac{z}{x^2}(\log z)^2$

Dividing z on both sides, we get

$$\frac{1}{z}\frac{dz}{dx}+\frac{\log z}{x}=\frac{(\log z)^2}{x^2}$$

Put $\log z=y$

$$\frac{1}{z}\frac{dz}{dx}=\frac{dy}{dx}$$

Therefore, $\frac{dy}{dx}+\frac{1}{x}y=\frac{y^2}{x^2}$

Dividing y^2 on both sides, we get

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=\frac{1}{x^2} \quad \text{---(1)}$$

Put $y^{-1}=t \Rightarrow -1y^{-2}\frac{dy}{dx}=\frac{dt}{dx}$

$$y^{-2}\frac{dy}{dx}=-\frac{dt}{dx}$$

(1) becomes, $-\frac{dt}{dx}+\frac{1}{x}t=\frac{1}{x^2}$

$$\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2}, \text{ this is linear in } t$$

$$P = -\frac{1}{x}, \qquad Q = -\frac{1}{x^2}$$

$$I.F = e^{\int p\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Therefore, the solution of the given equation is

$$t(I.F) = \int Q(I.F)\,dx + c$$

$$t\left(\frac{1}{x}\right) = \int -\frac{1}{x^2}\left(\frac{1}{x}\right)dx + c$$

$$y^{-1}\left(\frac{1}{x}\right) = -\int x^{-3}\,dx + c$$

$$\frac{1}{xy} = -\frac{x^{-2}}{-2} + c$$

$$\frac{1}{xy} = \frac{1}{2x^2} + c$$

$$\frac{1}{x\log z} = \frac{1}{2x^2} + c$$

Exercises

1) $x\frac{dy}{dx} - 2y = xy^4$

2) $\frac{dy}{dx} + xy = xy^3$

3) $\frac{dy}{dx} - \frac{y}{x} = \frac{1}{2y}$

4) $\frac{dy}{dx} - 2y\tan x = y^2 \tan^2 x$

5) $\frac{dy}{dx} - \frac{2y}{x} = \left(\frac{y}{x}\right)^3$

6) $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

7) $\frac{dy}{dx} + y\cos x = y^n \sin 2x$

8) $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$

9) $x(x-y)\,dy + y^2\,dx = 0$

10) $\frac{dy}{dx} - y\tan x = \frac{\sin x}{y^2 \sec^2 x}$

11) $\frac{dy}{dx} = x^3y^2 + xy$

12) $\frac{dy}{dx} + xy = x^3y^3$

13) $x\frac{dy}{dx} + y = y^2 \log x$

14) $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$

15) $\frac{dy}{dx} + xy = y^2 e^{\frac{x^2}{2}} \log x$

Answers

1) $-y^{-3}x^6 = \frac{3}{7}x^7 + c$

2) $\frac{1}{y^2} = 1 + cx^{x^2}$

3) $y^2 = -x + cx^2$

4) $\frac{\sec^2 x}{y} = \frac{\tan^3 x}{3} + c$

5) $(x^2+y^2)x^2 = cy^2$

6) $-\frac{e^{x^2}}{y^2} = c - 2x$

7) $y^{1-n} = 2\left[\sin x + \frac{1}{n-1}\right] + ce^{n-1}\sin x$

8) $\frac{1}{\log z} = \frac{1}{2x} + cx$

9) $\frac{y}{x} = \log y + c$

10) $y^3 \cos^3 x = c - \left(\frac{1}{2}\right)\cos^6 x$

11) $\frac{1}{y} = x^2 - 2 + ce^{-\frac{x^2}{2}}$

12) $\frac{1}{y^2} = (1+x^2) + ce^{x^2}$

13) $y(1 + cx + \log x) = 1$

14) $(x+y)^{-2} = (x^2+1)ce^{x^2}$

15) $y(x - x\log x + c) = e^{-\frac{x^2}{2}}$

EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

Such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is called an exact differential equation.

To solve an exact differential equation we should remember the following steps.

Step 1 Verify $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Step 2 Integrate M with respect to x keeping y as constant.

Step 3 Integrate the terms in N those are free from x, with respect to y.

Step 4 Add step (2) and step (3) and equate to an arbitrary constant, which is the required solution i.e.,

$$\underbrace{\int M(x, y)dx}_{\text{treating } y \text{ as constant}} + \underbrace{\int N(x, y)dy}_{\text{terms in } N \text{ which are free from } x} = c$$

Worked Examples

1. *Solve* $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2 y + 2x\right)dy = 0$

➡ $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2 y + 2x\right)dy = 0$

$M = \left(2x^3 - xy^2 - 2y + 3\right)$ $\quad N = -\left(x^2 y + 2x\right)$

$\therefore \dfrac{\partial M}{\partial y} = -2xy - 2$ $\quad \dfrac{\partial N}{\partial x} = -2xy - 2$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, the given equation is an exact equation.

$\therefore$ The general solution of the given equation is

$$\underbrace{\int \left(2x^3 - xy^2 - 2y + 3\right)dx}_{\text{treating } y \text{ as a constant}} - \underbrace{\int \left(x^2 y + 2x\right)dy}_{\text{the terms do not contain } x} = c$$

$$2\frac{x^4}{4} - \frac{x^2}{2}y^2 - 2xy + 3x = c$$

$$\frac{1}{2}x^4 - \frac{1}{2}x^2 y^2 - 2xy + 3x = c$$ ■

2. *Solve* $\dfrac{dy}{dx}=\dfrac{x^2-2xy}{x^2-\sin y}$

➡ $\dfrac{dy}{dx}=\dfrac{x^2-2xy}{x^2-\sin y}$

This equation can be written as

$$(x^2-\sin y)dy=(x^2-2xy)dx$$
$$(x^2-2xy)dx-(x^2-\sin y)dy=0$$
$$M=x^2-2xy \qquad N=-(x^2-\sin y)$$
$$\frac{\partial M}{\partial y}=-2x \qquad \frac{\partial N}{\partial x}=-2x$$
$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Therefore, the given equation is exact, and the solution is

$$\int(x^2-2xy)dx \qquad -\int(x^2-\sin y)dy=c$$

treating y as constant the terms do no contain x

$$\frac{x^3}{3}-2\frac{x^2}{2}y-(0-(-\cos y))=c$$
$$\frac{x^3}{3}-x^2y-\cos y=c$$

3. *Solve* $(x^2+y)dx+(y^3+x)dy=0$

➡ $(x^2+y)dx+(y^3+x)dy=0$

$$M=x^2+y \qquad N=y^3+x$$
$$\frac{\partial M}{\partial y}=1 \qquad \frac{\partial N}{\partial x}=1$$
$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Therefore, the given equation is exact and the solution is

$$\int(x^2+y)dx+\int(y^3+x)dy=c$$

$$\frac{x^3}{3} + xy + \frac{y^3}{3} = c$$
$$x^3 + y^3 + 3xy = 3c$$
$$x^3 + y^3 + 3xy = c_1 \qquad ■$$

4. *Solve* $(e^y + y\cos xy)dx + (xe^y + x\cos xy)dy = 0$

➡ $(e^y + y\cos xy)dx + (xe^y + x\cos xy)dy = 0$

$$M = e^y + y\cos xy \qquad N = xe^y + x\cos xy$$
$$\frac{\partial M}{\partial y} = e^y + y(-\sin xy.\,x) + 1.\cos xy$$
$$= e^y - xy\sin xy + \cos xy$$
$$\frac{\partial N}{\partial x} = e^y + x(-\sin xy.\;y) + 1.\cos xy$$
$$= e^y - xy\sin xy + \cos xy$$
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore, the given equation is exact and the solution is

$$\int (e^y + y\cos xy)dx + \int (xe^y + x\cos xy)dy = c$$
$$xe^y + y\frac{\sin(xy)}{y} = c$$
$$\therefore xe^y + \sin xy = c \qquad ■$$

5. *Solve* $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

➡ $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

$$M = (2xy + y - \tan y) \qquad N = x^2 - x\tan^2 y + \sec^2 y$$
$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y \qquad \frac{\partial N}{\partial x} = 2x - \tan^2 y$$
$$= 2x - (\sec^2 y - 1)$$
$$= 2x - \sec^2 y + 1$$
$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore, the given equation is exact and the solution is given by,

$$\int (2xy + y - \tan y)dx \quad + \quad \int (x^2 - x\tan^2 y + \sec^2 y)dx = c$$

treating y as constant the terms do no contain x

$$2\frac{x^2}{2}y + yx - x\tan y + \tan y = c$$

$$x^2 y + xy - x\tan y + \tan y = c$$ ■

6. *Solve* $y\sin 2x dx - (1 + y + \cos^2 x)dy = 0$

➡ $y\sin 2x dx - (1 + y + \cos^2 x)dy = 0$

$M = y\sin 2x$ $\qquad N = -(1 + y + \cos^2 x)$

$\frac{\partial M}{\partial y} = \sin 2x$ $\qquad \frac{\partial N}{\partial x} = -[2\cos x(-\sin x)]$

$$\frac{\partial N}{\partial x} = 2\sin x\cos x$$

$$\frac{\partial N}{\partial x} = \sin 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore, the given equation is exact and the solution is,

$$\int y\sin 2x\, dx - \int (1 + y + \cos^2 x)dy = c$$

$$y\left(-\frac{\cos 2x}{2}\right) - \left(y + \frac{y^2}{2}\right) = c$$

$$y\cos 2x + 2y + y^2 = c_1, \text{ where } c_1 = -2c$$ ■

7. *Solve* $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

➡ $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

$M = y\cos x + \sin y + y$ $\qquad N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ $\qquad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$