

$$= 4x^2 \left( \frac{\partial z}{\partial u} \right)^2 + 4y^2 \left( \frac{\partial z}{\partial v} \right)^2 + 8xy \frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v}$$

$$+ 4x^2 \left( \frac{\partial z}{\partial v} \right)^2 + 4y^2 \left( \frac{\partial z}{\partial u} \right)^2 - 8xy \frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v}$$

$$\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = 4(x^2 + y^2) \left[ \left( \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial v} \right)^2 \right]$$

(ii) Consider,

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 2x^2 \frac{\partial z}{\partial u} + 2xy \frac{\partial z}{\partial v} + 2y^2 \frac{\partial z}{\partial u} - 2xy \frac{\partial z}{\partial v}$$

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 2(x^2 + y^2) \frac{\partial z}{\partial u}$$

13. If  $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$  find the value of  $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z}$

$$u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$$

$$\text{Let } v = \frac{y-x}{xy} \text{ and } w = \frac{z-x}{xz}$$

$$v = \frac{1}{x} - \frac{1}{y}$$

$$w = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{x^2}$$

$$\frac{\partial w}{\partial x} = -\frac{1}{x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{y^2}$$

$$\frac{\partial w}{\partial y} = 0$$

$$\frac{\partial v}{\partial z} = 0$$

$$\frac{\partial w}{\partial z} = \frac{1}{z^2}$$

Therefore,  $u = u(v, w)$

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$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \left( -\frac{1}{x^2} \right) + \frac{\partial u}{\partial w} \left( -\frac{1}{x^2} \right)$$

Multiplying  $x^2$  on both sides, we get

$$x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \quad \text{---(1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \left( \frac{1}{y^2} \right) + \frac{\partial u}{\partial w} (0)$$

Multiplying  $y^2$  on both sides, we get

$$y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \quad \text{---(2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} (0) + \frac{\partial u}{\partial w} \frac{1}{z^2}$$

Multiplying  $z^2$  on both sides, we get

$$z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \quad \text{---(3)}$$

Adding equations (1), (2) and (3), we get

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0 \quad \blacksquare$$

### Exercises

- 1) If  $u = x^4 y^5$ , where  $x = t^2$  and  $y = t^3$ , find  $\frac{du}{dt}$
- 2) If  $x^3 + 3x^2 y + 6xy^2 = 1$ , find  $\frac{dy}{dx}$

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- 3) If  $u = x^2 - y^2 + \sin yz$ , where  $y = e^x$ , find  $\frac{du}{dx}$
- 4) If  $u = x \log xy$  and  $x^3 + y^2 - 3axy = 1$ , find  $\frac{du}{dx}$
- 5) Find  $\frac{du}{dx}$  if  $u = \sin(x^2 + y^2)$ , where  $a^2x^2 + b^2y^2 = e^2$  and  $y = v + e^{-v}$
- 6) If  $u = x \log xy$ , where  $x^3 + y^3 + 3xy = 1$ , find  $\frac{du}{dx}$
- 7) Find  $\frac{dy}{dx}$  of  $y^x + x^y = e$
- 8) If  $u = \tan^{-1}\left(\frac{y}{x}\right)$ , where  $x = e^t - e^{-t}$  and  $y = e^t + e^{-t}$ , find  $\frac{du}{dt}$
- 9) If  $u = xy^2 + x^2y$ , where  $x = at^2$  and  $y = 2at$ , find  $\frac{du}{dt}$
- 10) Find  $\frac{du}{dt}$  if  $u = e^y \sin x$ ,  $x = t^2$ ,  $y = \log t$
- 11) Find  $\frac{dy}{dx}$  if  $xe^{-y} - 2ye^x = 1$
- 12) Find  $\frac{du}{dx}$  if  $u = e^{(x^2+y^2)}$ ,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- 13) If  $z = f(x, y)$  and  $x = e^u$  and  $y = e^{-u} - e^v$ , show that  $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$
- 14) If  $z = f(u, v)$  where  $u = x^2 - y^2$  and  $v = 2xy$ , prove that  $x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 2(x^2 + y^2) \frac{\partial z}{\partial u}$
- 15) If  $f$  is a homogeneous function of  $x, y, z$  with degree  $n$ , prove that  $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf$

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16) If  $u = x + y$ ,  $v = xy$ ,  $f = f(u, v)$ , show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = u \frac{\partial f}{\partial u} + 2v \frac{\partial f}{\partial v}$$

17) If  $z = \log(u^2 + v)$ ,  $u = e^{x^2+y^2}$ ,  $x^2 + y^2$ , show that

$$\frac{\partial z}{\partial x} = \frac{2x(2u^2 + 1)}{u^2 + v}, \quad \frac{\partial z}{\partial y} = \frac{2yu^2 + 1}{u^2 + v}$$

18) If  $u = x^2$ ,  $x = 2r - 3s$ ,  $y = -r + 3s - 5$ , prove that  $\frac{\partial u}{\partial r} = 4x + 2y$

19) If  $z = f(x, y)$  and  $u = ax + by$ ,  $v = ay - bx$ , prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (a^2 + b^2) \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$$

20) If  $u$  and  $v$  are functions of  $x$  and  $y$  defined by  $x = u + e^{-v} \sin u$ , and

$$y = v + e^{-v}, \text{ prove that } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$$

### Answers

1)  $\frac{du}{dt} = 23t^{2z}$

2)  $\frac{dy}{dx} = \frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}$

3)  $\frac{du}{dx} = 2x + (-2y + z \cos yz)e^x + \frac{y \cos yz}{x}$

4)  $\frac{du}{dx} = 1 + \log xy + \{x(x^2 - ay)\} / \{y(ax - y^2)\}$

5)  $\frac{du}{dx} = 2x \{ \cos(x^2 + y^2) \} \left( 1 - \frac{a^2}{b^2} \right)$

6)  $\frac{du}{dx} = 1 + \log xy - x(x^2 + y) / y(x + y^2)$

7)  $\frac{du}{dx} = -(y^x \log y + yx^{y-1}) / (xy^{x-1} + x^y \log x)$

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8)  $\frac{du}{dt} = -\frac{2}{e^{2t} + e^{-2t}}$

9)  $\frac{du}{dt} = y^3 + 2xy^2 + yaxy + 2ax^2$

10)  $\frac{du}{dt} = \sin t^2 + 2t^2 \cos t^2$

11)  $\frac{dy}{dx} = \frac{e^{-y} - 2ye^x}{xe^{-y} + 2e^x}$

12)  $\frac{du}{dx} = \frac{2x}{a^2} (a^2 - b^2) e^{(x^2+y^2)}$

**JACOBIANS**

Let  $u$  and  $v$  are functions of two independent variables  $x$  and  $y$ , then *Jacobian* of  $u$  and  $v$  with respect to  $x$  and  $y$  is denoted by

$$J \text{ or } \frac{\partial(u,v)}{\partial(x,y)} \text{ or } J \begin{pmatrix} u, v \\ x, y \end{pmatrix}$$

and is defined by

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

If  $u, v, w$  are functions of three independent variables  $x, y, z$ , then

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

**Properties of Jacobian**

- 1) If  $u$  and  $v$  are the functions of  $s$  and  $t$ , and  $s$  and  $t$  are the functions of  $x$  and  $y$ , then

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$$J = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(x,y)}$$

**Proof**  $\frac{\partial(u,v)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{vmatrix} \begin{vmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} & \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y} \\ \frac{\partial v}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial v}{\partial t} \frac{\partial t}{\partial x} & \frac{\partial v}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial v}{\partial t} \frac{\partial t}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

Therefore  $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(x,y)}$

**Inverse Jacobians**

If  $J = \frac{\partial(u,v)}{\partial(x,y)}$ , then the *inverse Jacobian* of the Jacobian  $J$  is denoted

and defined by  $J' = \frac{\partial(x,y)}{\partial(u,v)}$

2) Prove that  $JJ' = 1$

**Proof**  $JJ' = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u} & \frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial u} & \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial v} \end{vmatrix}$$

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$$= \begin{vmatrix} \frac{\partial u}{\partial v} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial v} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

**Null or Zero Jacobian**

A Jacobian  $J$  is said to be *null* or *zero Jacobian* if  $J=0$

**Remark** If  $u = u(x, y)$  and  $v = v(x, y)$  are mutually independent, then the Jacobian  $J \neq 0$ . If  $u$  and  $v$  are not independent or functionally dependent, then Jacobian  $J=0$

**Worked Examples**

1. If  $u = e^x \cos y$ ,  $v = e^x \sin y$  find  $\frac{\partial(u, v)}{\partial(x, y)}$

$$\begin{aligned} \rightarrow \quad u &= e^x \cos y & v &= e^x \sin y \\ \frac{\partial u}{\partial x} &= e^x \cos y & \frac{\partial v}{\partial x} &= e^x \sin y \\ \frac{\partial u}{\partial y} &= -e^x \sin y & \frac{\partial v}{\partial y} &= e^x \cos y \end{aligned}$$

$$\begin{aligned} \therefore J &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} \\ &= e^{2x} \cos^2 y + e^{2x} \sin^2 y \\ &= e^{2x} (\cos^2 y + \sin^2 y) = e^{2x} \end{aligned}$$

2. If  $u = 1 - x$ ,  $v = x(1 - y)$ ,  $w = xy(1 - z)$  find  $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

$$\begin{aligned} \rightarrow \quad u &= 1 - x & v &= x(1 - y) & w &= xy(1 - z) \\ \frac{\partial u}{\partial x} &= -1 & \frac{\partial v}{\partial x} &= (1 - y) & \frac{\partial w}{\partial x} &= y(1 - z) \end{aligned}$$

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$$\begin{aligned} \frac{\partial u}{\partial y} &= 0 & \frac{\partial v}{\partial y} &= -x & \frac{\partial w}{\partial y} &= x(1-z) \\ \frac{\partial u}{\partial z} &= 0 & \frac{\partial v}{\partial z} &= 0 & \frac{\partial w}{\partial z} &= -xy \end{aligned}$$

$$\begin{aligned} \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 1-y & -x & 0 \\ y(1-z) & x(1-z) & -xy \end{vmatrix} \\ &= -x^2y \end{aligned}$$

3. If  $u = x^2 + y^2 + z^2$ ,  $v = xy + yz + zx$ ,  $w = x + y + z$ , find

$$J \frac{(u, v, w)}{(x, y, z)}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x & \frac{\partial v}{\partial x} &= y+z & \frac{\partial w}{\partial x} &= 1 \\ \frac{\partial u}{\partial y} &= 2y & \frac{\partial v}{\partial y} &= x+z & \frac{\partial w}{\partial y} &= 1 \\ \frac{\partial u}{\partial z} &= 2z & \frac{\partial v}{\partial z} &= y+x & \frac{\partial w}{\partial z} &= 1 \end{aligned}$$

$$J \frac{(u, v, w)}{(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 2x(x+z-y-x) - 2y(y+z-y-x) + 2z(y+z-x-z) \\ &= 2x(z-y) - 2y(z-x) + 2z(y-x) \\ &= 2xz - 2xy - 2yz + 2xy + 2zy - 2xz \\ &= 0 \end{aligned}$$

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**Note** We observe that  $w = x + y + z$

$$\begin{aligned}w^2 &= (x + y + z)^2 \\ &= x^2 + y^2 + z^2 + 2(xy + yz + zx) \\ w^2 &= u + 2v\end{aligned}$$

Hence, we got  $J = 0$

4. If  $u = x + y + z$ ,  $y = v + z$ ,  $z = uvw$ , find Jacobian of  $x, y, z$  with respect to  $u, v, w$ .

$$\begin{aligned}\rightarrow z &= uvw \\ v &= y + z \\ y &= v - z = v - uvw \\ u &= x + y + z \\ x &= u - (y + z) = u - v\end{aligned}$$

$$\begin{aligned}x &= u - v & y &= v - uvw & z &= uvw \\ \frac{\partial x}{\partial u} &= 1 & \frac{\partial y}{\partial u} &= -vw & \frac{\partial z}{\partial u} &= vw \\ \frac{\partial x}{\partial v} &= -1 & \frac{\partial y}{\partial v} &= 1 - uw & \frac{\partial z}{\partial v} &= uw \\ \frac{\partial x}{\partial w} &= 0 & \frac{\partial y}{\partial w} &= -uv & \frac{\partial z}{\partial w} &= uv\end{aligned}$$

$$\begin{aligned}\frac{\partial(x, y, z)}{\partial(u, v, w)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ vw & 1 - uw & -uv \\ vw & uw & uv \end{vmatrix} \\ &= 1[(1 - uw)uv + uwwv] + 1(-vwuv + vwuv) \\ &= uv - u^2vw + u^2vw = uv\end{aligned}$$

5. If  $u = x + y$ ,  $v = y + z$ ,  $w = z + x$ . Find  $J \frac{(u, v, w)}{(x, y, z)}$

$$\rightarrow \therefore u = x + y \Rightarrow \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = 0$$

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$$v = y + z \Rightarrow \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 1, \quad \frac{\partial v}{\partial z} = 1$$

$$w = z + x \Rightarrow \frac{\partial w}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 1$$

$$J \frac{(u, v, w)}{(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 1(0-1) + 0(0-1) = 1 + 1 + 0 = 2 \quad \blacksquare$$

6. If  $u = \frac{yz}{x}$ ,  $v = \frac{zx}{y}$ ,  $w = \frac{xy}{z}$ , show that  $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

$$\begin{aligned} u = \frac{yz}{x} &\Rightarrow \frac{\partial u}{\partial x} = \frac{yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial u}{\partial z} = \frac{y}{x} \\ v = \frac{zx}{y} &\Rightarrow \frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial v}{\partial y} = -\frac{zx}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{x}{y} \\ w = \frac{xy}{z} &\Rightarrow \frac{\partial w}{\partial x} = \frac{y}{z}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}, \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2} \end{aligned}$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2} \left( -\frac{zx}{y^2} \cdot \left( -\frac{xy}{z^2} \right) - \frac{y}{y} \cdot \left( \frac{x}{z} \right) \right) - \frac{z}{x} \left( \frac{z}{y} \cdot \left( -\frac{xy}{z^2} \right) - \frac{x}{y} \cdot \left( \frac{y}{z} \right) \right)$$

$$+ \frac{y}{x} \left( \frac{z}{y} \cdot \left( \frac{x}{z} \right) - \left( -\frac{zx}{y^2} \right) \cdot \frac{y}{z} \right)$$

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$$\begin{aligned}
 &= -\frac{yz}{x^2} \left( \frac{x^2}{yz} - \frac{x^2}{yz} \right) - \frac{z}{x} \left( -\frac{x}{z} - \frac{x}{z} \right) + \frac{y}{x} \left( \frac{x}{y} + \frac{x}{y} \right) \\
 &= -\frac{yz}{x^2} (0) - \frac{z}{x} \left( -2\frac{x}{z} \right) + \frac{y}{x} \left( 2\frac{x}{y} \right) = 0 + 2 + 2 = 4 \quad \blacksquare
 \end{aligned}$$

7. If  $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \theta$  then find

$$\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)}$$

$$\begin{aligned}
 \Rightarrow x &= r \sin \theta \cos \phi & y &= r \sin \theta \sin \phi & z &= r \cos \theta \\
 \frac{\partial x}{\partial r} &= \sin \theta \cos \phi & \frac{\partial y}{\partial r} &= \sin \theta \sin \phi & \frac{\partial z}{\partial r} &= \cos \theta \\
 \frac{\partial x}{\partial \theta} &= r \cos \theta \cos \phi & \frac{\partial y}{\partial \theta} &= r \cos \theta \sin \phi & \frac{\partial z}{\partial \theta} &= -r \sin \theta \\
 \frac{\partial x}{\partial \phi} &= -r \sin \theta \sin \phi & \frac{\partial y}{\partial \phi} &= r \sin \theta \cos \phi & \frac{\partial z}{\partial \phi} &= 0
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\
 &= \begin{vmatrix} \sin \theta \cos \phi & -r \sin \theta \sin \phi & r \cos \theta \cos \phi \\ \sin \theta \sin \phi & r \sin \theta \cos \phi & r \cos \theta \sin \phi \\ \cos \theta & 0 & -r \sin \theta \end{vmatrix} \\
 &= \sin \theta \cos \phi [-r^2 \sin^2 \theta \cos \phi - 0] \\
 &\quad + r \sin \theta \sin \phi [-r \sin^2 \theta \sin \phi - r \cos^2 \theta \sin \phi] \\
 &\quad + r \cos \theta \cos \phi [0 - r \sin \theta \cos \theta \cos \phi] \\
 &= -r^2 \sin^3 \theta \cos^2 \phi - r^2 \sin^3 \theta \sin^2 \phi - r^2 \sin \theta \cos^2 \theta \sin^2 \phi \\
 &\quad + 0 - r^2 \sin \theta \cos^2 \theta \cos^2 \phi
 \end{aligned}$$

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$$\begin{aligned}
 &= -r^2 \sin^3 \theta [\cos^2 \phi + \sin^2 \phi] - r^2 \sin \theta \cos^2 \theta [\sin^2 \phi + \cos^2 \phi] \\
 &= -r^2 \sin^3 \theta (1) - r^2 \sin \theta \cos^2 \theta (1) \\
 &= -r^2 \sin \theta (\sin^2 \theta + \cos^2 \theta) \\
 &= -r^2 \sin \theta (1) \\
 \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} &= -r^2 \sin \theta
 \end{aligned}$$

8. If  $u = x(1-y)$ ,  $v = xy$ , find  $J = \frac{\partial(u, v)}{\partial(x, y)}$  and  $J' = \frac{\partial(x, y)}{\partial(u, v)}$

verify that  $JJ' = 1$

$$\rightarrow \begin{aligned} u &= x(1-y) & v &= xy \end{aligned}$$

$$\frac{\partial u}{\partial x} = 1-y \quad \frac{\partial v}{\partial x} = y$$

$$\frac{\partial u}{\partial y} = -x \quad \frac{\partial v}{\partial y} = x$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = (1-y)x + xy = x$$

$$u = x - xy \quad v = xy$$

$$u = x - v \quad y = v/x$$

$$x = u + v \quad y = \frac{v}{u+v}$$

$$\frac{\partial x}{\partial u} = 1 \quad \frac{\partial y}{\partial u} = \frac{-v}{(u+v)^2}$$

$$\frac{\partial x}{\partial v} = 1 \quad \frac{\partial y}{\partial v} = \frac{(u+v) - v}{(u+v)^2} = \frac{u}{(u+v)^2}$$

$$J' = \begin{vmatrix} 1 & 1 \\ -v & u \end{vmatrix} \frac{1}{\begin{vmatrix} (u+v)^2 & (u+v)^2 \end{vmatrix}}$$

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$$= \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2}$$

$$= \frac{u+v}{(u+v)^2} = \frac{1}{u+v} = \frac{1}{x}$$

$$J.J' = x \cdot \frac{1}{x} = 1$$

9. If  $u = x^2 - 2y$ ,  $v = x + y$  find  $J$  and  $J'$ , verify that  $JJ' = 1$

$$\rightarrow u = x^2 - 2y \quad v = x + y$$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = -2 \quad \frac{\partial v}{\partial y} = 1$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2 \\ 1 & 1 \end{vmatrix} = 2x + 2 = 2(x+1)$$

$$u = x^2 - 2y \quad v = x + y \Rightarrow y = v - x$$

$$u = x^2 - 2(v - x)$$

$$u + 2(v - x) = x^2$$

$$x^2 + 2x - (u + 2v) = 0$$

It is quadratic equation in  $x$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-(u+2v))}}{2(1)} = \frac{-2 \pm 2\sqrt{1+u+2v}}{2}$$

$$x = -1 \pm \sqrt{1+u+2v}$$

$$\text{Let } x = -1 + \sqrt{1+u+2v} \quad \therefore y = v + 1 - \sqrt{1+u+2v}$$

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{1+u+2v}} \quad \frac{\partial y}{\partial u} = -\frac{1}{2\sqrt{1+u+2v}}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2\sqrt{1+u+2v}} \cdot 2 \quad \frac{\partial y}{\partial v} = 1 - \frac{1}{2\sqrt{1+u+2v}} \cdot 2$$

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$$\frac{\partial x}{\partial v} = \frac{1}{\sqrt{1+u+2v}} \quad \frac{\partial y}{\partial v} = 1 - \frac{1}{\sqrt{1+u+2v}}$$

$$J' = \begin{vmatrix} \frac{1}{2\sqrt{1+u+2v}} & \frac{1}{\sqrt{1+u+2v}} \\ -1 & 1 \end{vmatrix} = \frac{1}{2\sqrt{1+u+2v}} \left( 1 - \frac{1}{\sqrt{1+u+2v}} \right) + \frac{1}{2(1+u+2v)}$$

$$J' = \frac{1}{2\sqrt{1+u+2v}} = \frac{1}{2(x+1)}$$

$$\therefore JJ' = (2x+2) \frac{1}{2(x+1)} = 2(x+1) \frac{1}{2(x+1)} = 1$$

10. If  $x = e^u \sec v$ ,  $y = e^u \tan v$ , prove that  $JJ' = 1$

$$\Rightarrow x = e^u \sec v \quad y = e^u \tan v$$

$$\frac{\partial x}{\partial u} = e^u \sec v \quad \frac{\partial y}{\partial u} = e^u \tan v$$

$$\frac{\partial x}{\partial v} = e^u \cdot \sec v \cdot \tan v \quad \frac{\partial y}{\partial v} = e^u \sec^2 v$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \sec v & e^u \sec v \tan v \\ e^u \tan v & e^u \sec^2 v \end{vmatrix}$$

$$= e^{2u} \sec^3 v - e^{2u} \sec v \tan^2 v$$

$$= e^{2u} \sec v (\sec^2 v - \tan^2 v)$$

$$= e^{2u} \sec v$$

From the given equations, we have

$$\frac{y}{x} = \frac{e^u \tan v}{e^u \sec v} = \sin v \quad x^2 - y^2 = e^{2u} \sec^2 v - e^{2u} \tan^2 v$$

$$\therefore v = \sin^{-1}(y/x) \quad x^2 - y^2 = e^{2u} \Rightarrow u = \frac{1}{2} \log(x^2 - y^2)$$

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$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 - y^2} \cdot (2x) = \frac{x}{x^2 - y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 - y^2} \cdot (-2y) = -\frac{y}{x^2 - y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1 - (y/x)^2}} \left( -\frac{y}{x^2} \right) = \frac{-y}{x\sqrt{x^2 - y^2}}$$

$$\frac{\partial v}{\partial y} = \frac{1}{\sqrt{1 - (y/x)^2}} \left( \frac{1}{x} \right) = \frac{1}{\sqrt{x^2 - y^2}}$$

$$\begin{aligned} J' = \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2 - y^2} & \frac{y}{x^2 - y^2} \\ \frac{-y}{x\sqrt{x^2 - y^2}} & \frac{1}{\sqrt{x^2 - y^2}} \end{vmatrix} \\ &= \frac{x}{x^2 - y^2} \cdot \frac{1}{\sqrt{x^2 - y^2}} - \frac{y}{x^2 - y^2} \cdot \frac{-y}{x\sqrt{x^2 - y^2}} \\ &= \frac{1}{(x^2 - y^2)^{\frac{3}{2}}} - \frac{y^2}{x(x^2 - y^2)^{\frac{3}{2}}} = \frac{x^2 - y^2}{x(x^2 - y^2)^{\frac{3}{2}}} \\ &= \frac{1}{x\sqrt{x^2 - y^2}} \end{aligned}$$

Substituting for  $x$  and  $x^2 - y^2$ , we get

$$J' = \frac{1}{e^u \sec v \cdot \sqrt{e^{2u}}} = \frac{1}{e^{2u} \sec v}$$

$$\therefore JJ' = e^{2u} \sec v \cdot \frac{1}{e^{2u} \sec v} = 1 \quad \blacksquare$$

11. If  $u = x + y$ ,  $v = xy$ , verify that  $JJ' = 1$ .

$$\rightarrow u = x + y, \quad v = xy$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = y, \quad \frac{\partial v}{\partial y} = x$$

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