*Prove that*

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta$$

***Proof***

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta\,d\theta$

If $x=0\Rightarrow\theta=0$, if $x=1\Rightarrow\theta=\pi/2$

$$\therefore\beta(m,n)=\int_0^{\pi/2}\left(\sin^2\theta\right)^{m-1}\left(1-\sin^2\theta\right)^{n-1}2\sin\theta\cos\theta\,d\theta$$

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

We know that $\beta(m,n)=\beta(n,m)$

$$\therefore\beta(n,m)=2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta$$

$$\therefore\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta$$

*Prove that*

$$\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=2\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=2\int_0^{\pi/2}\sin^q\theta\cos^p\theta\,d\theta$$

***Proof*** We have,

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta$$

Put $2m-1=p$ $\quad 2n-1=q$

$2m=p+1$ $\quad 2n=q+1$

$m=\dfrac{p+1}{2}$ $\quad n=\dfrac{q+1}{2}$

$$\therefore\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=2\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=2\int_0^{\pi/2}\sin^q\theta\cos^p\theta\,d\theta$$

***Note*** (1) If $q=0$, then

$$\beta\left(\frac{p+1}{2},\frac{1}{2}\right)=2\int_0^{\frac{\pi}{2}}\sin^p\theta\,d\theta=2\int_0^{\frac{\pi}{2}}\cos^p\theta\,d\theta$$

If $p=0,\ q=0$, then

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=2\int_0^{\frac{\pi}{2}}d\theta=\pi$$

(2) (i) $\int_0^{\pi/2}\sin^p\theta\cos^q\theta d\theta=\int_0^{\pi/2}\sin^q\theta\cos^p\theta d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

(ii) $\int_0^{\pi/2}\sin^p\theta d\theta=\int_0^{\pi/2}\cos^p\theta d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{1}{2}\right)$

***Relation between Beta and Gamma Functions***

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

***Proof*** We know that $\Gamma(n)=2\int_0^{\infty}e^{-x^2}x^{2n-1}dx$

Similarly $\Gamma(m)=2\int_0^{\infty}e^{-y^2}y^{2m-1}dy$

$$\therefore\Gamma(m)\Gamma(n)=\left[2\int_0^{\infty}e^{-y^2}y^{2m-1}dy\right]\left[2\int_0^{\infty}e^{-x^2}x^{2n-1}dx\right]$$

$$=4\int_0^{\infty}\int_0^{\infty}e^{-\left(x^2+y^2\right)}x^{2n-1}y^{2m-1}dxdy$$

$x:0\to\infty$
$y:0\to\infty$

Now, we use polar coordinates to evaluate the above integral

i.e., $x=r\cos\theta$ and $y=r\sin\theta\Rightarrow dxdy=rdrd\theta$

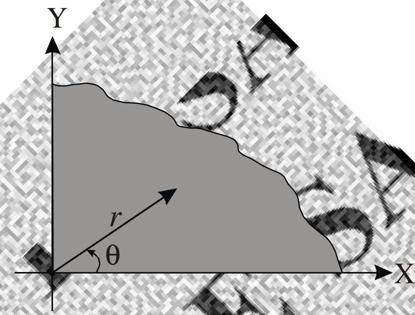


**Figure 3.11**

Here $\theta : 0 \to \frac{\pi}{2}$ and $r : 0 \to \infty$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1}\, r\,dr\,d\theta$$

$$= 4\int_{0}^{\pi/2}\int_{0}^{\infty} e^{-r^2} r^{2n-1+2m-1+1}\cos^{2n-1}\theta\sin^{2m-1}\theta\; dr\,d\theta$$

$$= 2\int_{0}^{\infty} e^{-r^2} r^{2(m+n)-1}\,dr \times 2\int_{0}^{\pi/2}\cos^{2n-1}\theta\sin^{2m-1}\theta\; d\theta$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

***Note*** $\beta(1-n,n) = \Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin n\pi}$

**Worked Examples**

1. *Evaluate the following*

(i) $\beta(4,5)$ (ii) $\beta\left(\frac{3}{2},2\right)$ (iii) $\beta\left(\frac{1}{4},\frac{3}{4}\right)$

➡ (i) $\beta(4,5) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(4+5)}$

$$= \frac{3!4!}{8!} = \frac{3!4!}{8\cdot 7\cdot 6\cdot 5\cdot 4!} = \frac{1}{280}$$

(ii) $$\beta\left(\frac{3}{2},2\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{\Gamma\left(\frac{3}{2}+2\right)} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma(2)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{4}{15}$$

(iii) $$\beta\left(\frac{1}{4},\frac{3}{4}\right) = \beta\left(\frac{1}{4},1-\frac{1}{4}\right) = \frac{\pi}{\sin\left(\pi\frac{1}{4}\right)} = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,\pi$$ ■

2. *Evaluate the following integrals*

(i) $\int_0^1 x^5(1-x)^6\,dx$ (ii) $\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}}\,dx$

(iii) $\int_0^1 \frac{x^3}{\sqrt{1-x}}\,dx$ (iv) $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx$

(v) $\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$

***Note*** $\int_0^1 x^m(1-x)^n\,dx = \beta(m+1,n+1)$

➡ (i) Let $I = \int_0^1 x^5(1-x)^6\,dx = \beta(5+1,6+1)$

$$= \beta(6,7) = \frac{\Gamma(6)\Gamma(7)}{\Gamma(6+7)} = \frac{5!6!}{12!}$$

$$= \frac{5.4.3.2.6!}{12.11.10.9.8.7.6!} = \frac{1}{5544}$$

(ii) $I = \int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}}dx$

Put $x = 4t \Rightarrow dx = 4dt$

If $x = 0 \Rightarrow t = 0$, if $x = 4 \Rightarrow t = 1$

$$I = \int_0^1 (4t)^{\frac{3}{2}}(4-4t)^{\frac{5}{2}}4dt$$

$$= \int_0^1 4^{\frac{3}{2}} \cdot t^{\frac{3}{2}} \cdot 4^{\frac{5}{2}} \cdot (1-t)^{\frac{5}{2}} \cdot 4dt$$

$$= 4^5 \int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{5}{2}}dt = 4^5 \ \beta\left(\frac{3}{2}+1, \frac{5}{2}+1\right)$$

$$= 4^5 \beta\left(\frac{5}{2}, \frac{7}{2}\right) = \frac{4^5 \Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{7}{2}\right)}$$

$$= \frac{4^5 \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{\Gamma(6)} = \frac{1440\sqrt{\pi}\sqrt{\pi}}{5!} = 12\pi$$

(iii) $\int_0^1 \frac{x^3}{\sqrt{1-x}}dx = \int_0^1 x^3(1-x)^{-\frac{1}{2}}dx$

$$= \beta\left(3+1, -\frac{1}{2}+1\right) = \beta\left(4, \frac{1}{2}\right)$$

$$= \frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)} = \frac{3!\sqrt{\pi}}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{3!\sqrt{\pi}}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)} = \frac{32}{35}$$

(iv) $\int_0^1 \sqrt{\frac{1-x}{x}}dx = \int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}}dx$

$$= \int_0^1 (1-x)^{\frac{1}{2}} x^{-\frac{1}{2}} dx = \beta\left(\frac{1}{2}+1, -\frac{1}{2}+1\right)$$

$$= \beta\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{1}{2}\sqrt{\pi}\sqrt{\pi} = \frac{\pi}{2}$$

(v) Let $I = \int_0^1 \frac{1}{\sqrt{1-x^4}}dx$

Put $x^4 = t \Rightarrow x = t^{\frac{1}{4}} \Rightarrow dx = \frac{1}{4}t^{\frac{1}{4}-1}dt \quad = \frac{1}{4}t^{\frac{-3}{4}}dt$

If $x = 0 \Rightarrow t = 0$, if $x = 1 \Rightarrow t = 1$

$$\therefore I = \int_0^1 \frac{1}{\sqrt{1-t}}\frac{1}{4}t^{\frac{-3}{4}}dt = \frac{1}{4}\int_0^1 (1-t)^{-\frac{1}{2}} t^{\frac{-3}{4}} dt$$

$$= \frac{1}{4}\beta\left(-\frac{1}{2}+1, -\frac{3}{4}+1\right) = \frac{1}{4}\beta\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\right)} = \frac{1}{4}\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4}\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

■

3. *Prove the following*

(i) $\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

(ii) $\beta(m,n)=\frac{1}{2^{m+n-1}}\int_0^1 (1+x)^{m-1}(1-x)^{n-1}dx$

(iii) $\beta(m,n)=2a^m b^n\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta}{\left(a\sin^2\theta+b\cos^2\theta\right)^{m+n}}d\theta$

➡ (i) We know that $\beta(m,n)=\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx$

$$\beta(m,n)=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx+\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \text{---(1)}$$

Let $I=\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Put $x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}dt$

If $x=1\Rightarrow t=1$, if $x=\infty\Rightarrow t=0$

$$\therefore I=\int_1^0 \frac{(1/t)^{m-1}}{(1+(1/t))^{m+n}}\left(-\frac{1}{t^2}dt\right)$$

$$=-\int_1^0 \frac{1}{t^{m-1}\left(\frac{t+1}{t}\right)^{m+n}}\cdot\frac{1}{t^2}dt$$

$$=\int_0^1 \frac{1}{t^{m-1-m-n+2}(1+t)^{m+n}}dt$$

$$I=\int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}dt=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx \qquad \text{---(2)}$$

Substituting (2) in (1), we get

$$\beta(m,n)=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx+\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

(ii) We have, $\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Put $x=\frac{1+t}{1-t}$

$$dx=\frac{(1-t)1-(1+t)(-1)}{(1-t)^2}\,dx=\frac{2}{(1-t)^2}$$

and $1+x=1+\frac{1+t}{1-t}=\frac{1-t+1+t}{1-t}=\frac{2}{1-t}$

If $x=0\Rightarrow t=-1$, if $x=\infty\Rightarrow t=1$

$$\therefore \beta(m,n)=\int_{-1}^{1}\frac{\left(\frac{1+t}{1-t}\right)^{m-1}}{\left(\frac{2}{1-t}\right)^{m+n}}\frac{2dt}{(1-t)^2}$$

$$=\int_{-1}^{1}\frac{(1+t)^{m-1}}{(1-t)^{m-1}}\cdot\frac{(1-t)^{m+n}}{2^{m+n}}\cdot\frac{2dt}{(1-t)^2}$$

$$=\frac{1}{2^{m+n-1}}\int_{-1}^{1}(1+t)^{m-1}(1-t)^{m+n-m+1-2}\,dt$$

$$\beta(m,n)=\frac{1}{2^{m+n-1}}\int_{-1}^{1}(1+t)^{m-1}(1-t)^{n-1}\,dt$$

(iii) We have, $\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Put $x=\frac{a}{b}\tan^2\theta$

$$dx = \frac{a}{b} 2 \tan\theta \sec^2\theta \, d\theta$$

If $x = 0 \Rightarrow \theta = 0$, if $x = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \beta(m,n) = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{a}{b}\tan^2\theta\right)^{m-1}}{\left(1+\frac{a}{b}\tan^2\theta\right)^{m+n}} 2\frac{a}{b}\tan\theta\sec^2\theta\, d\theta$$

$$= 2\left(\frac{a}{b}\right)^m \int_0^{\frac{\pi}{2}} \frac{\tan^{2m-2}\theta\tan\theta\sec^2\theta}{\left(1+\frac{a}{b}\frac{\sin^2\theta}{\cos^2\theta}\right)^{m+n}} d\theta$$

$$= 2\left(\frac{a}{b}\right)^m \int_0^{\frac{\pi}{2}} \frac{\tan^{2m-1}\theta\sec^2\theta\, d\theta}{\frac{\left(b\cos^2\theta + a\sin^2\theta\right)^{m+n}}{b^{m+n}\cos^{2m+2n}\theta}}$$

$$= 2a^m b^n \int_0^{\frac{\pi}{2}} \frac{\cos^{2m+2n}\theta \frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta}\cdot\frac{1}{\cos^2\theta}}{\left(a\sin^2\theta + b\cos^2\theta\right)^{m+n}} d\theta$$

$$\beta(m,n) = 2a^m b^n \int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta}{\left(a\sin^2\theta + b\cos^2\theta\right)} d\theta \quad ■$$

4. *Show that* $\beta(m,n)\beta(m+n,p) = \beta(n,p)\beta(n+p,m)$

➡ $$\beta(m,n)\beta(m+n,p) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\cdot\frac{\Gamma(m+n)\Gamma(p)}{\Gamma(m+n+p)}$$

$$= \frac{\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(m+n+p)} \quad \text{---(1)}$$

$$\beta(n,p)\beta(n+p,m) = \frac{\Gamma(n)\Gamma(p)}{\Gamma(n+p)}\cdot\frac{\Gamma(n+p)\Gamma(m)}{\Gamma(n+p+m)}$$

$$= \frac{\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(m+n+p)} \qquad \text{---(2)}$$

From equations (1) and (2), we have

$$\therefore\ \beta(m,n)\beta(m+n,p) = \beta(n,p)\beta(n+p,m) \qquad \blacksquare$$

5. *Prove that* $\dfrac{\beta(m+1,n)}{m} = \dfrac{\beta(m,n+1)}{n} = \dfrac{\beta(m,n)}{m+n}$

➡ We know that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$\therefore\ \beta(m+1,n) = \int_0^1 x^{m}(1-x)^{n-1}\,dx$$

Applying integration by parts, we get

$$= \left[x^m \frac{(1-x)^n}{-n}\right]_0^1 - \int_0^1 m x^{m-1}\left(\frac{(1-x)^n}{-n}\right)dx$$

$$= \frac{m}{n}\int_0^1 x^{m-1}(1-x)^n\,dx$$

$$\beta(m+1,n) = \frac{m}{n}\,\beta(m,n+1)$$

$$\Rightarrow \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = k(\text{say})$$

$$\therefore\ k = \frac{\beta(m+1,n)+\beta(m,n+1)}{m+n} \qquad \because \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$$

$$= \frac{1}{m+n}\left\{\int_0^1 x^m(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^n\,dx\right]$$

$$= \frac{1}{m+n}\left\{\int_0^1 x^{m-1}(1-x)^{n-1}\left[x+(1-x)\right]dx\right.$$

$$= \frac{1}{n+m}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{1}{m+n}\,\beta(m,n)$$

$$k = \frac{\beta\,(m,n)}{m+n}$$

$$\therefore \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n} \qquad \blacksquare$$

6. *Show that* $\dfrac{\beta(m+2,n-2)}{\beta(m,n)} = \dfrac{m(m+1)}{(n-1)(n-2)}$

➡ We have $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m,n)}$ ---(1)

$$\beta(m+2,n-2) = \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m+2+n-2)} = \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m+n)} \quad \text{---(2)}$$

$$\therefore \frac{\beta(m+2,n-2)}{\beta(m,n)} = \frac{\dfrac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m+n)}}{\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$$= \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m)\Gamma(n)}$$

$$= \frac{\Gamma(\overline{m+1}+1)\Gamma(n-2)}{\Gamma(m)\Gamma(\overline{n-1}+1)} = \frac{(m+1)\Gamma(m+1)\Gamma(n-2)}{\Gamma(m)(n-1)\Gamma(n-1)}$$

$$= \frac{(m+1)\Gamma(m+1)\Gamma(n-2)}{\Gamma(m)(n-1)\Gamma(n-2+1)}$$

$$= \frac{(m+1)m\Gamma(m)\Gamma(n-2)}{\Gamma(m)(n-1)\Gamma(\overline{n-2}+1)}$$

$$= \frac{m(m+1)\Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)}$$

$$\frac{\beta(m+2,n-2)}{\beta(m,n)} = \frac{m(m+1)}{(n-1)(n-2)} \qquad \blacksquare$$