$$\frac{dy}{dx} + 2y\tan x = \sin x, \text{ this is linear in } y$$

$P = 2\tan x \qquad Q = \sin x$

$$\text{I.F} = e^{\int p dx} = e^{\int 2\tan x dx} = e^{2\log(\sec x)} = e^{\log(\sec^2 x)} = \sec^2 x$$

Therefore, the solution of the given equation is

$$y(I.F) = \int Q(I.F)dx + c$$

$$y\sec^2 x = \int \sin x \cdot \sec^2 x dx + c$$

$$= \int \tan x \cdot \sec x dx + c$$

$$y\sec^2 x = \sec x + c$$

Given $y = 0$ when $x = \pi/3$

Therefore, $(0)\sec^2(\pi/3) = \sec(\pi/3) + c$

$0 = 2 + c \Rightarrow c = -2$

Therefore, the solution of the given equation is

$$y\sec^2 x = \sec x - 2$$

17. *Solve* $\frac{dy}{dx} = (\sin x - \sin y)\frac{\cos x}{\cos y}$

➡ $\frac{dy}{dx} = (\sin x - \sin y)\frac{\cos x}{\cos y}$

$$\cos y\frac{dy}{dx} = (\sin x - \sin y)\cos x$$

$$\cos y\frac{dy}{dx} + \sin y\cos x = \sin x\cos x$$

Put $\sin y = z \Rightarrow \cos y \ \frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dz}{dx} + z\cos x = \sin x\cos x, \text{ this is linear in } z$$

$P = \cos x \qquad Q = \sin x\cos x$

$$\text{I.F} = e^{\int p dx} = e^{\int \cos x dx} = e^{\sin x}$$

Therefore, the solution of the given equation is

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$ze^{\sin x} = \int \sin x \cos x e^{\sin x}\,dx + c$$

Put $\sin x = t \Rightarrow \cos x dx = dt$

$$ze^{\sin x} = \int te^t\,dt + c$$

$$\sin y e^{\sin x} = te^t - \int 1 \cdot e^t\,dt + c = te^t - e^t + c$$

$$= (t-1)e^t + c$$

$$\sin y e^{\sin x} = (\sin x - 1)e^{\sin x} + c$$

$$\sin y = \sin x - 1 + ce^{-\sin x}$$

Exercises

1) $(x + \tan y)\,dy = \sin 2y\,dx$

2) $\frac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$

3) $\frac{dy}{dx} + y\tan x = \sec x$

4) $(x^2 - 1)\frac{dy}{dx} + 2xy = 1$

5) $\frac{dy}{dx} - y\cot x = 2x\sin x$

6) $\frac{dy}{dx} - \frac{2y}{x} = x^2\sin 3x$

7) $(\cos^3 x)y' + y\cos x = \sin x$

8) $y' + y\cos x = \sin 2x$

9) $\sqrt{1-y^2}\,dx = (\sin^{-1} y - x)\,dy$

10) $e^y\left(\frac{dy}{dx} + 1\right) = e^x$

11) $y' + y\sec x = \tan x$

12) $y' + y\cot x = \cos x$

13) $\tan y\frac{dy}{dx} + \tan x = \cos y\cos^3 x$

14) $x\frac{dy}{dx} + y\log y = xye^x$

15) $\frac{dy}{dx} = \frac{e^{-2\sqrt{x}} - y}{\sqrt{x}}$

16) $\frac{dy}{dx} + 3x^2 y = x^5 e^{x^3}$

17) $\cos^2 x\frac{dy}{dx} + y = \tan x$

18) $y' - y\tan x = e^x\sec x$

19) $xy' - y = x^5\log x$

20) $(1 - x^2)y' + 2xy = x\sqrt{1-x^2}$ given that $y = 0$ when $x = 0$

Answers

1) $x = \tan y + c\sqrt{\tan y}$

2) $y \sin x = 2x^2 + c$

3) $y = (\tan x + c)\cos x$

4) $y(1 - x^2) = c - x$

5) $y \operatorname{cosec} x = x^2 + c$

6) $x^2 \cos 3x + 3y = cx^2$

7) $1 + y = \tan x - ce^{-\tan x}$

8) $y + 2 = 2\sin x + ce^{-\sin x}$

9) $x = \sin^{-1} y - 1 + ce^{-\sin^{-1} y}$

10) $e^y = \frac{1}{2}e^x + ce^{x}$

11) $(y-1)(1+\sin x) = (c - x)\cos x$

12) $2y \sin x - \sin^2 x = c$

13) $\sec y \sec x = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

14) $x \log y = (x-1)e^x + c$

15) $ye^{2\sqrt{x}} = 2\sqrt{x} + c$

16) $12y = e^{x^3}(2x^3 - 1) + ce^{-x^3}$

17) $y + 1 = \tan x + ce^{-\tan x}$

18) $y \cos x = e^x + c$

19) $16y = x^4(4\log x - 1) + cx$

20) $y(1-x^2)^{-1} = (2-x^2)^{-1/2} - 1$

BERNOULLI'S EQUATION

An equation of the form

$$\frac{dy}{dx} + P(x)y = y^n Q(x) \qquad \text{---(1)}$$

is called Bernoulli's equation and the other form is

$$\frac{dx}{dy} + P(y)x = Q(y)x^n \qquad \text{---(2)}$$

If $n = 0$, then (1) and (2) are linear D E, for other values of n, (1) and (2) not linear D E, but they can be transformed to linear D E by small transformation.

Dividing y^n on both sides of equation (1); we get

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

Put $z = y^{1-n}$ i.e., $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$

Therefore, (1) becomes, $\frac{dz}{dx}+(1-n)P(x)z=(1-n)Q(x)$

which is a linear D E.

Similarly, we can reduce equation (2) to linear D.E by dividing x^n on both sides.

Worked Examples

1. *Solve* $\frac{dy}{dx}+\frac{y}{x}=y^2x$

➡ $\frac{dy}{dx}+\frac{y}{x}=y^2x$

This is a Bernoulli's equation, dividing by y^2 on both sides, we get

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=x \qquad ---(1)$$

Put $y^{-1}=t \Rightarrow -1y^{-2}\frac{dy}{dx}=\frac{dt}{dx}$

$$y^{-2}\frac{dy}{dx}=-\frac{dt}{dx}$$

$\therefore$ (1) becomes, $-\frac{dt}{dx}+\frac{1}{x}t=x$

i.e., $\frac{dt}{dx}-\frac{1}{x}t=-x$, this is linear in *t*.

Here $P=-\frac{1}{x}$ $\qquad Q=-x$

I.F. $=e^{\int Pdx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

Therefore, the solution of the given equation is

$$t(I.F.)=\int Q(I.F.)dx+c$$

$$y^{-1}\left(\frac{1}{x}\right)=\int(-x)\frac{1}{x}dx+c$$

$$\frac{1}{xy} = -x + c$$

$$\frac{1}{xy} + x = c$$ ■

2. *Solve* $x\frac{dy}{dx} + y = x^3 y^6$

➡ $x\frac{dy}{dx} + y = x^3 y^6$, dividing by xy^6 on both sides, we get

$$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2 \quad \text{---(1)}$$

Put $y^{-5} = t \Rightarrow -5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$

$$y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$\therefore$ (1) becomes, $-\frac{1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2$

i.e., $\frac{dt}{dx} - \frac{5}{x}t = -5x^2$, this is linear in t.

Here $P = -\frac{5}{x}$, $Q = -5x^2$

$$\therefore \text{ I.F.} = e^{\int Pdx} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$$

Therefore, the solution of the given equation is

$$t(I.F.) = \int Q(I.F.)\,dx + c$$

$$y^{-5}\left(\frac{1}{x^5}\right) = \int -5x^2\left(\frac{1}{x^5}\right)dx + c$$

$$\frac{1}{x^5y^5} = -5\int x^{-3}dx + c = -5\frac{x^{-2}}{-2} + c$$

$$\frac{1}{x^5y^5} = \frac{5}{2x^2} + c$$ ■

3. *Solve* $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$

➡ $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$, dividing by $y^4 x^3$ on both sides, we get

$$y^{-4} \frac{dy}{dx} - \frac{1}{x} y^{-3} = -\frac{\cos x}{x^3} \qquad \text{---(1)}$$

Put $y^{-3} = t \Rightarrow -3y^{-4} \frac{dy}{dx} = \frac{dt}{dx}$

$$y^{-4} \frac{dy}{dx} = -\frac{1}{3} \frac{dt}{dx}$$

$\therefore$ (1) becomes, $-\frac{1}{3} \frac{dt}{dx} - \frac{1}{x} t = -\frac{\cos x}{x^3}$

i.e., $\frac{dt}{dx} + \frac{3}{x} t = \frac{3\cos x}{x^3}$, this is linear in t.

Here $P = \frac{3}{x}, Q = \frac{3\cos x}{x^3}$

$$I.F. = e^{\int \frac{3}{x} dx} = e^{3 \log x} = e^{\log x^3} = x^3$$

Therefore, the solution of the given equation is

$$t(I.F.) = \int Q(I.F.)dx + c$$

$$y^{-3}(x^3) = \int \left(\frac{3\cos x}{x^3} \right) x^3 dx + c$$

$$\frac{x^3}{y^3} = 3\sin x + c$$

■

4. *Solve* $(x^2 y^3 + xy)\frac{dy}{dx} = 1$

➡ $(x^2 y^3 + xy)\frac{dy}{dx} = 1$

This equation can be written as

$$x^2 y^3 + xy = \frac{dx}{dy}$$

$\frac{dx}{dy} - xy = x^2y^3$, dividing by x^2 on both sides, we get

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \qquad \text{---(1)}$$

Put $x^{-1} = t \Rightarrow -1x^{-2}\frac{dx}{dy} = \frac{dt}{dy}$

$$x^{-2}\frac{dx}{dy} = -\frac{dt}{dy}$$

$\therefore$ (1) becomes , $-\frac{dt}{dx} - yt = y^3$

i.e., $\frac{dt}{dx} + yt = -y^3$, this is linear in t.

Here $P = y$, $Q = -y^3$

$$I.F. = e^{\int Pdy} = e^{\int ydy} = e^{\frac{y^2}{2}}$$

Therefore, the solution of the given equation is

$$t(I.F.) = \int Q(I.F.)dy + c$$

$$x^{-1}e^{\frac{y^2}{2}} = \int -y^3 e^{\frac{y^2}{2}} dy + c$$

$$\frac{e^{\frac{y^2}{2}}}{x} = -\int y^3 e^{\frac{y^2}{2}} dy + c$$

Put $\frac{y^2}{2} = t \Rightarrow ydy = dt$

$$\therefore \frac{e^{\frac{y^2}{2}}}{x} = -\int y^2 e^{\frac{y^2}{2}}.ydy + c$$

$$= -\int 2te^t dt + c = -2\left[te^t - \int 1.e^t\right] + c$$

$$= -2te^t + 2e^t + c$$

$$\frac{e^{\frac{y^2}{2}}}{x} = -y^2 e^{\frac{y^2}{2}} + 2e^{\frac{y^2}{2}} + c$$

or $1 = -xy^2 + 2x + cxe^{-\frac{y^2}{2}}$ ■

5. *Solve* $y\left(x^2 y + e^x\right)dx = e^x dy$

➡ $y\left(x^2 y + e^x\right)dx = e^x dy$

$$y\left(x^2 y + e^x\right) = e^x \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y\left(x^2 y + e^x\right)}{e^x}$$

$$\frac{dy}{dx} = x^2 y^2 e^{-x} + y$$

$\frac{dy}{dx} - y = x^2 y^2 e^{-x}$, dividing by y^2 on both sides, we get

$$y^{-2}\frac{dy}{dx} - y^{-1} = x^2 e^{-x} \qquad \text{---(1)}$$

Put $y^{-1} = t \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$

$$y^{-2}\frac{dy}{dx} = \frac{-dt}{dx}$$

∴ The equation (1) gives, $\frac{-dt}{dx} - t = x^2 e^{-x}$

$\frac{dt}{dx} + t = -x^2 e^{-x}$, this is linear in *t*.

Here $P = 1$, $Q = -x^2 e^{-x}$

$$I.F = e^{\int P dx} = e^{\int 1.dx} = e^{x}$$

∴ The solution of the given equation is,

$$t(I.F.) = \int Q(I.F.)dx + c$$

$$y^{-1}\left(e^x\right) = \int -x^2 e^{-x}.e^x dx + c$$