6. *If u be a homogeneous function of degree n, show that*

(i) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = (n-1)\dfrac{\partial u}{\partial x}$

(ii) $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}$

7. *If* $u = xyf\left(\dfrac{x}{y}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

8. *If* $u = x^3 y^2 \sin^{-1}\left(\dfrac{y}{x}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 5u$

9. *If* $u = \sin^{-1}\left(\dfrac{x^3 - y^3}{x - y}\right)$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u.$

10. *If* $u = e^{x^3 + y^3}$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u \log u.$

11. *If* $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

12. *If* $u = \sin^{-1}\left(\dfrac{x^2 y^2}{x + y}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u.$

13. *If* $u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u$

14. *If* $u = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$

15. *If* $u = \tan^{-1}\left(\dfrac{x^3 y^3}{x^3 + y^3}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3}{2}\sin 2u$

16. *If* $u = \tan^{-1}\sqrt{x^3y + xy^3 + y^4 + x^4}$, *prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

17. *If* $u = x^4y^6\cos^{-1}\left(\frac{y}{x}\right)$, *prove that*

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 10u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 9u$

18. *If* $u = \sin^{-1}\left(\frac{3x^2 + 4y^2}{3x + 4y}\right)$, *prove that*

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u$

19. *If* $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

20. *If* $u = \sqrt{x^2 - y^2}\,\sin^{-1}\left(\frac{y}{x}\right)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

21. *If* $u = \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4u$

22. *If* $u = \frac{x^3 + y^3 + z^3}{xyz + x^2y + y^2z}$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

23. *If* $u = \sin^{-1}\left(\frac{x^3 + y^3 + z^3}{\sqrt{x + y + z}}\right)$, *prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$$

24. *If* $u = \cos^{-1}\left(\dfrac{x^3 + y^3 + z^3}{\sqrt{x^4 + y^4 + z^4}}\right)$, *prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\cot u$$

25. *If* $u = x\phi\left(\dfrac{y}{x}\right) + \phi\left(\dfrac{x}{y}\right)$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Total Differential and Total Derivative

If $f = f(x, y)$ be a function of two independent variables x and y, then the total differential is defined as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If $f = f(x, y)$ be a function such that $x = x(t)$ and $y = y(t)$ then we can find the value of f in terms of t by substituting from the last two equations in the first equation. Hence we can regard f as a function of the single variable t, and we can find the ordinary differentiation or the total derivative df/dt by using

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

Similarly, if $f = f(x_1, x_2, x_3 ...., x_n)$ then total differential

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_n}dx_n$$

and if $f = f(x_1, x_2, x_3 ...., x_n)$ and $x_1, x_2, ..., x_n$ are all functions of t, then total derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dt} + ... + \frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dt}$$

Differentiation of implicit functions

If $f(x, y)$ is a function of *x* and *y*, and *y* is a function of *x*, then differentiation of *f* with respect to *x* is

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

Differentiation of Composite functions

Let $f(x, y)$ be a function of *x* and *y*, where *x* and *y* are themselves functions of two other variables *u* and *v*, then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

and $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Similarly, if *f* is a function of *u* and *v*, where *u* and *v* are themselves functions of *x* and *y*, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

Worked Examples

1. *If* $f = e^x[x\sin y + y\cos y]$ *find the total differential.*

➡ $f = e^x[x\sin y + y\cos y]$

$$\frac{\partial f}{\partial x} = e^x[\sin y] + e^x[x\sin y + y\cos y] = e^x[(1+x)\sin y + y\cos y]$$

$$\frac{\partial f}{\partial y} = e^x[x\cos y + y(-\sin y) + 1\cdot\cos y] = e^x[(x+1)\cos y - y\sin y]$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Rightarrow df = e^x[(1+x)\sin y + y\cos y]dx + e^x[(x+1)\cos y - y\sin y]dy$$ ■

2. *Find the total differential of* $u = x^2 y + y^2 z + z^2 x$

➡ $$\frac{\partial u}{\partial x} = 2xy + z^2, \quad \frac{\partial u}{\partial y} = 2yz + x^2, \quad \frac{\partial u}{\partial z} = 2zx + y^2$$

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy + \frac{\partial u}{\partial z}.dz$$

$$du = \left(2xy + z^2\right)dx + \left(2yz + x^2\right)dy + \left(2zx + y^2\right)dz$$ ■

3. *Find the total differentiation of* $u = x^3 y^2$, *where* $x = e^t$, $y = \log t$

➡ $$u = x^3 y^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{du}{dt} = \left(3x^2 y^2\right)e^t + \left(2x^3 y\right)\frac{1}{t}$$

$$= 3e^{2t}(\log t)^2.e^t + 2e^{3t}\log t\cdot\frac{1}{t}$$

$$= e^{3t}\log t\left[3\log t + \frac{2}{t}\right]$$ ■

4. *If* $u = x^2 + y^2 + z^2$ *where* $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$, *show that without actually substituting that* $\frac{du}{dt} = 4e^{2t}$

➡ $u = x^2 + y^2 + z^2$

$x = e^t$ $\quad y = e^t \sin t$ $\quad z = e^t \cos t$

$$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = e^t \cos t + e^t \sin t \qquad \frac{dz}{dt} = e^t(-\sin t) + e^t \cos t$$

$$\frac{dy}{dt} = e^t(\cos t + \sin t) \qquad \frac{dz}{dt} = e^t(\cos t - \sin t)$$

We have $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2x\left(e^t\right) + 2ye^t(\cos t + \sin t) + 2ze^t(\cos t - \sin t)$$

$$= 2e^t\left[x + y(\cos + \sin t) + z(\cos t - \sin t)\right]$$

$$= 2e^t\left[e^t + e^t \sin t(\cos t + \sin t) + e^t \cos t(\cos t - \sin t)\right]$$

$$= 2e^{2t}\left[1 + \sin t \cos t + \sin^2 t + \cos^2 t - \cos t \sin t\right]$$

$$= 2e^{2t}[1+1]$$

$$\frac{du}{dt} = 4e^{2t}$$ ■

5. *If* $z = z(x, y)$ *and* $x = u - v,\ y = uv$, *show that*

(i) $\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = (u+v)\frac{\partial z}{\partial y}$

(ii) $u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = (u+v)\frac{\partial z}{\partial x}$

➡ $x = u - v$ $\qquad$ $y = u.v$

$\frac{\partial x}{\partial u} = 1$ $\qquad$ $\frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = -1$ $\qquad$ $\frac{\partial y}{\partial v} = u$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}(v) \quad \text{---(1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \quad \text{---(2)}$$

(i) Adding equations (1) and (2), we get

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} = (u+v)\frac{\partial z}{\partial y}$$

(ii) Taking $u\times(1) - v\times(2)$, we get

$$u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v\frac{\partial z}{\partial x} - uv\frac{\partial z}{\partial y} = (u+v)\frac{\partial z}{\partial x}$$ ■

6. *If* $z = f(x, y)$ *and* $x = e^u \cos v,\ y = e^u \sin v$, *show that*

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}$$

➡ $x = e^u \cos v \qquad y = e^u \sin v,$

$$\frac{\partial x}{\partial u} = e^u \cos v \qquad \frac{\partial y}{\partial u} = e^u \sin v$$

$$\frac{\partial x}{\partial v} = -e^u \sin v \qquad \frac{\partial y}{\partial v} = e^u \cos v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = e^u \cos v \frac{\partial z}{\partial x} + e^u \sin v \frac{\partial z}{\partial y}$$

$$= e^u\left[\cos v \frac{\partial z}{\partial x} + \sin v \frac{\partial z}{\partial y}\right] \qquad \text{---(1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = -e^u \sin v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial z}{\partial y}$$

$$= e^u\left[\cos v \frac{\partial z}{\partial y} - \sin v \frac{\partial z}{\partial x}\right] \qquad \text{---(2)}$$

Squaring and adding equations (1) and (2), we get

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\cos v \frac{\partial z}{\partial x} + \sin v \frac{\partial z}{\partial y}\right]^2$$

$$+ e^{2u}\left[\cos v \frac{\partial z}{\partial y} - \sin v \frac{\partial z}{\partial x}\right]^2$$

$$= e^{2u}\left[\cos^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2 v\left(\frac{\partial z}{\partial y}\right)^2 + 2\cos v \frac{\partial z}{\partial x}.\sin v \frac{\partial z}{\partial y}\right.$$

$$\left. + \cos^2 v\left(\frac{\partial z}{\partial y}\right)^2 + \sin^2 v\left(\frac{\partial z}{\partial x}\right)^2 - 2\cos v \frac{\partial z}{\partial y}.\sin v \frac{\partial z}{\partial x}\right]$$

$$= e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2(\cos^2 v+\sin^2 v)+\left(\frac{\partial z}{\partial y}\right)^2(\sin^2 v+\cos^2 v)\right]$$

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$ ■

7. *If $u = f(x, y)$ where $x = r\cos\theta,\ \ y = r\sin\theta$, show that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

➡ $x = r\cos\theta$ $\qquad y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y} \qquad \text{---(1)}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y} \qquad \text{---(2)}$$

Therefore, $\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

$$= \left(\cos\theta\cdot\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right)^2+\frac{1}{r^2}\left(r\cos\theta\frac{\partial u}{\partial y}-r\sin\theta\frac{\partial u}{\partial x}\right)^2$$

$$= \cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2+\sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2+2\cos\theta\cdot\sin\theta\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}$$

$$+\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2+\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2-2\cos\theta\cdot\sin\theta\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2(\sin^2\theta+\cos^2\theta)+\left(\frac{\partial u}{\partial y}\right)^2(\sin^2\theta+\cos^2\theta)$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$ ■

8. *If* $u=f(x-y,y-z,z-x)$, *prove that* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

➡ Let $v=x-y \qquad t=y-z \qquad s=z-x$

$$\frac{\partial v}{\partial x}=1 \qquad \frac{\partial t}{\partial x}=0 \qquad \frac{\partial s}{\partial x}=-1$$

$$\frac{\partial v}{\partial y}=-1 \qquad \frac{\partial t}{\partial y}=1 \qquad \frac{\partial s}{\partial y}=0$$

$$\frac{\partial v}{\partial z}=0 \qquad \frac{\partial t}{\partial z}=-1 \qquad \frac{\partial s}{\partial z}=1$$

Therefore, $u=f(v,t,s)$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}=\frac{\partial u}{\partial v}-\frac{\partial u}{\partial s} \qquad \text{---(1)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}=-\frac{\partial u}{\partial v}+\frac{\partial u}{\partial t} \qquad \text{---(2)}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}=-\frac{\partial u}{\partial t}+\frac{\partial u}{\partial s} \qquad \text{---(3)}$$

Adding equations (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial v}-\frac{\partial u}{\partial s}-\frac{\partial u}{\partial v}+\frac{\partial u}{\partial t}-\frac{\partial u}{\partial t}+\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$ ■

9. *If* $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$, *prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

➡ Let $v=\frac{x}{y} \qquad t=\frac{y}{z} \qquad s=\frac{z}{x}$

$$\frac{\partial v}{\partial x} = \frac{1}{y} \qquad \frac{\partial t}{\partial x} = 0 \qquad \frac{\partial s}{\partial x} = -\frac{z}{x^2}$$

$$\frac{\partial v}{\partial y} = -\frac{x}{y^2} \qquad \frac{\partial t}{\partial y} = \frac{1}{z} \qquad \frac{\partial s}{\partial y} = 0$$

$$\frac{\partial v}{\partial z} = 0 \qquad \frac{\partial t}{\partial z} = -\frac{y}{z^2} \qquad \frac{\partial s}{\partial z} = \frac{1}{x}$$

Therefore, $u = f(v, t, s)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial v} - \frac{z}{x^2}\frac{\partial u}{\partial s} \quad \text{---(1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2}\frac{\partial u}{\partial v} + \frac{1}{z}\frac{\partial u}{\partial t} \quad \text{---(2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$\frac{\partial u}{\partial z} = -\frac{y}{z^2}\frac{\partial u}{\partial t} + \frac{1}{x}\frac{\partial u}{\partial s} \quad \text{---(3)}$$

Adding the equations $x\times(1)$, $y\times(2)$ and $z\times(3)$, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial v} - \frac{z}{x}\frac{\partial u}{\partial s} - \frac{x}{y}\frac{\partial u}{\partial v} + \frac{y}{z}\frac{\partial u}{\partial t} - \frac{y}{z}\frac{\partial u}{\partial t} + \frac{z}{x}\frac{\partial u}{\partial s}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

■

10. *If* $u = \frac{x}{z}, v = \frac{y}{z}, w = z$ *and* $f = f(u, v, w)$, *prove that*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = w\frac{\partial f}{\partial w}$$

➡ Let $u = \dfrac{x}{z}$ $\quad v = \dfrac{y}{z}$ $\quad w = z$

$$\frac{\partial u}{\partial x} = \frac{1}{z} \qquad \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = \frac{1}{z} \qquad \frac{\partial w}{\partial y} = 0$$

$$\frac{\partial u}{\partial z} = -\frac{x}{z^2} \qquad \frac{\partial v}{\partial z} = -\frac{y}{z^2} \qquad \frac{\partial w}{\partial z} = 1$$

Therefore, $f = f(u, v, w)$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{1}{z}\frac{\partial f}{\partial u} \qquad \text{---(1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{1}{z}\frac{\partial f}{\partial v} \qquad \text{---(2)}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z} = -\frac{x}{z^2}\frac{\partial f}{\partial u} - \frac{y}{z^2}\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \qquad \text{---(3)}$$

Adding the equations $x\times(1)$, $y\times(2)$, and $z\times(3)$ we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = \frac{x}{z}\frac{\partial f}{\partial u} + \frac{y}{z}\frac{\partial f}{\partial v} - \frac{x}{z}\frac{\partial f}{\partial u} - \frac{y}{z}\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} z$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = z\frac{\partial f}{\partial w}$$ ■

11. *If* $z = f(u, v)$, $u = x^2 - y^2$, $v = 2xy$ *show that*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4(x^2 + y^2)\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right]$$

➡ Let $u = x^2 - y^2$ $\quad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

Therefore, $z = f(u,v)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = 2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}$$

Differentiating partially with respect to *x*, we get

$$\frac{\partial^2 z}{\partial x^2} = 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + 2\frac{\partial z}{\partial u} + 2y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= 2x\left[\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}\right] + 2\frac{\partial z}{\partial u} + 2y\left[\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right]$$

$$= 2x\left[2x\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\partial v}\right] + 2\frac{\partial z}{\partial u} + 2y\left[2x\frac{\partial^2 z}{\partial u\partial v} + 2y\frac{\partial^2 z}{\partial v^2}\right]$$

$$= 4x^2\frac{\partial^2 z}{\partial u^2} + 8xy\frac{\partial^2 z}{\partial u\partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + z\frac{\partial z}{\partial u} \qquad \text{---(1)}$$

Differentiating *z* partially with respect to *y*, we get

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = -2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}$$

Differentiating again partially with respect to *y*, we get

$$\frac{\partial^2 z}{\partial y^2} = -2\frac{\partial z}{\partial u} - 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + 2x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= -2\frac{\partial z}{\partial u} - 2y\left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial y}\right] + 2x\left[\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial u}\right]$$

$$= -2\frac{\partial z}{\partial u} - 2y\left[-2y\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u\partial v}\right] + 2x\left[-2y\frac{\partial^2 z}{\partial u\partial v} + 2x\frac{\partial^2 z}{\partial v^2}\right]$$

$$= -2\frac{\partial z}{\partial u} + 4y^2\frac{\partial^2 z}{\partial u^2} - 8xy\frac{\partial^2 z}{\partial u\partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} \qquad \text{---(2)}$$

Adding equations (1) and (2), we get

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=4x^2\frac{\partial^2 z}{\partial u^2}+8xy\frac{\partial^2 z}{\partial u\partial v}+4y^2\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial u}+4y^2\frac{\partial^2 z}{\partial u^2}$$

$$-8xy\frac{\partial^2 z}{\partial u\partial v}+4x^2\frac{\partial^2 z}{\partial v^2}-2\frac{\partial z}{\partial u}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=4\left(x^2+y^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$$ ■

12. *If* $z=f(u,v)$, $u=x^2-y^2$, $v=2xy$ *, then prove the following*

(i) $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4\left(x^2+y^2\right)\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$

(ii) $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2\left(x^2+y^2\right)\frac{\partial z}{\partial u}$

➡ Let $u=x^2-y^2$ $\qquad v=2xy$

$\frac{\partial u}{\partial x}=2x$ $\qquad \frac{\partial v}{\partial x}=2y$

$\frac{\partial u}{\partial y}=-2y$ $\qquad \frac{\partial v}{\partial y}=2x$

Therefore, $z=f(u,v)$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=2x\frac{\partial z}{\partial u}+2y\frac{\partial z}{\partial v} \qquad \text{---(1)}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=-2y\frac{\partial z}{\partial u}+2x\frac{\partial z}{\partial v} \qquad \text{---(2)}$$

Squaring and adding equations (1) and (2), we get

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(2x\frac{\partial z}{\partial u}+2y\frac{\partial z}{\partial v}\right)^2+\left(2x\frac{\partial z}{\partial v}-2y\frac{\partial z}{\partial u}\right)^2$$