$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$ ■

2. *Evaluate the following*

(i) $\dfrac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$ (ii) $\dfrac{\Gamma(8)}{\Gamma(6)}$ (iii) $\dfrac{\Gamma\left(\frac{11}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$

➡

(i) $$\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{3}{2}+1\right)}{\sqrt{\pi}} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}} = \frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{\sqrt{\pi}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}} = \frac{\frac{3}{4}\sqrt{\pi}}{\sqrt{\pi}} = \frac{3}{4}$$

(ii) $$\frac{\Gamma(8)}{\Gamma(6)} = \frac{7!}{5!} = \frac{7.6.5!}{5!} = 42$$

(iii) $$\Gamma\left(\frac{11}{3}\right) = \Gamma\left(\frac{8}{3}+1\right) = \frac{8}{3}\Gamma\left(\frac{8}{3}\right) = \frac{8}{3}\Gamma\left(\frac{5}{3}+1\right)$$

$$= \frac{8}{3}\frac{5}{3}\Gamma\left(\frac{5}{3}\right) = \frac{40}{9}\Gamma\left(\frac{2}{3}+1\right)$$

$$= \frac{40}{9}\cdot\frac{2}{3}\Gamma\left(\frac{2}{3}\right) = \frac{80}{27}\Gamma\left(\frac{2}{3}\right)$$

$$\therefore \frac{\Gamma\left(\frac{11}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{80}{27}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{80}{27}$$ ■

3. *Evaluate the following*

(i) $\int_0^\infty x^5 e^{-x} dx$ (ii) $\int_0^\infty x^6 e^{-3x} dx$ (iii) $\int_0^\infty x^2 e^{-x^2} dx$

(iv) $\int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}} dx$ (v) $\int_0^\infty x^{\frac{3}{2}} e^{-4x} dx$ (vi) $\int_0^1 (\log x)^6 dx$

(vii) $\int_0^1 (x\log x)^3 dx$ (viii) $\int_0^1 \frac{1}{\sqrt{\log(1/x)}} dx$ (ix) $\int_0^1 \left(\log \frac{1}{x}\right)^{-\frac{3}{2}} dx$

(x) $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$

➡ (i) Let $I = \int_0^\infty x^5 e^{-x} dx$ Since $\int_0^\infty x^n e^{-x} dx = \Gamma(n+1)$

$= \Gamma(5+1) = 5!$

(ii) Let $I = \int_0^\infty x^6 e^{-3x} dx$

Put $3x = t \Rightarrow x = t/3 \Rightarrow dx = dt/3$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore I = \int_0^\infty \left(\frac{t}{3}\right)^6 e^{-t} \frac{dt}{3} = \frac{1}{3^7}\int_0^\infty t^6 e^{-t} dt = \frac{1}{3^7}\Gamma(6+1) = \frac{6!}{3^7}$$

(iii) Let $I = \int_0^\infty x^2 e^{-x^2} dx$

Put $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{1}{2\sqrt{t}} dt$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$I = \int_0^\infty t\, e^{-t} \frac{1}{2\sqrt{t}} dt = \int_0^\infty \sqrt{t}\, e^{-t} \frac{dt}{2}$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{1}{2}} e^{-t} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{\pi}$$

(iv) Let $I = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx$

Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore I = \int_0^\infty \left(t^2\right)^{\frac{1}{4}} e^{-t}\, 2t\,dt = 2\int_0^\infty t^{\frac{1}{2}} e^{-t}\, t\,dt$$

$$= 2\int_0^\infty t^{\frac{3}{2}}\, e^{-t}\, dt = 2\Gamma\left(\frac{3}{2}+1\right) \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= 2\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = 3\Gamma\left(\frac{1}{2}+1\right) = 3\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3\sqrt{\pi}}{2}$$

(v) Let $I = \int_0^\infty x^{\frac{3}{2}} e^{-4x}\, dx$

Put $4x = t \Rightarrow x = \frac{t}{4} \Rightarrow dx = \frac{dt}{4}$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore I = \int_0^\infty \left(\frac{t}{4}\right)^{\frac{3}{2}} e^{-t}\, \frac{dt}{4} = \frac{1}{4^{3/2}4}\int_0^\infty t^{\frac{3}{2}} e^{-t}\, dt$$

$$= \frac{1}{2^3 4}\Gamma\left(\frac{3}{2}+1\right) = \frac{1}{32}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{3}{64}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{64}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{128}$$

(vi) Let $I = \int_0^1 (\log x)^6\, dx$

Put $\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}\, dt$

If $x = 0 \Rightarrow t = \infty$, if $x = 1 \Rightarrow t = 0$

$$I=\int_{\infty}^{0}(-t)^6\left[-e^{-t}dt\right]$$

$$=-\int_{\infty}^{0}t^6e^{-t}dt=\int_{0}^{\infty}t^6e^{-t}dt \qquad \left[\because \int_a^b f(x)dx=-\int_b^a f(x)dx\right]$$

$$=\Gamma(6+1)=6!=720$$

(vii) Let $I=\int_0^1 (x\log x)^3 dx=\int_0^1 x^3(\log x)^3 dx$

Put $\log x=-t \Rightarrow x=e^{-t} \Rightarrow dx=-e^{-t}dt$

If $x=0\Rightarrow t=\infty$, if $x=1\Rightarrow t=0$

$$\therefore I=\int_{\infty}^{0}\left(e^{-t}\right)^3(-t)^3\left(-e^{-t}dt\right)$$

$$=\int_{\infty}^{0}e^{-4t}t^3dt=-\int_0^{\infty}e^{-4t}t^3dt$$

Put $4t=z\Rightarrow t=\frac{z}{4}\Rightarrow dt=\frac{dz}{4}$

If $t=0\Rightarrow z=0$, if $t=\infty\Rightarrow z=\infty$

$$I=-\int_0^{\infty}e^{-z}\left(\frac{z}{4}\right)^3\frac{dz}{4}$$

$$=\frac{-1}{4^4}\int_0^{\infty}z^3e^{-z}dz=-\frac{1}{4^4}\Gamma(3+1)$$

$$=\frac{-3!}{4^4}=\frac{-6}{256}=\frac{-3}{128}$$

(viii) Let $I=\int_0^1\frac{1}{\sqrt{\log(1/x)}}dx$

Put $\log\left(\frac{1}{x}\right)=t\Rightarrow\frac{1}{x}=e^t\Rightarrow x=\frac{1}{e^t}\Rightarrow x=e^{-t}\Rightarrow dx=-e^{-t}dt$

If $x=0\Rightarrow t=\infty$, if $x=1\Rightarrow t=0$

$$\therefore I = \int_{\infty}^{0} \frac{1}{\sqrt{t}}\left(-e^{-t}dt\right) = \int_{0}^{\infty} t^{\frac{-1}{2}} e^{-t} dt$$

$$= \Gamma\left(\frac{-1}{2}+1\right) = \Gamma\left(\frac{1}{2}\right)$$

$$I = \sqrt{\pi}$$

(ix) Let $I = \int_{0}^{1}\left(\log\frac{1}{x}\right)^{\frac{-3}{2}} dx$

Put $\log\frac{1}{x} = t \Rightarrow \frac{1}{x} = e^{t} \Rightarrow x = e^{-t}$

$dx = -e^{-t}dt$

If $x = 0 \Rightarrow t = \infty$, $x = 1 \Rightarrow t = 0$

$$\therefore I = \int_{\infty}^{0} t^{\frac{-3}{2}}\left(-e^{-t}dt\right) = \int_{0}^{\infty} t^{\frac{-3}{2}} e^{-t} dt$$

$$= \Gamma\left(\frac{-3}{2}+1\right) = \Gamma\left(\frac{-1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{-1}{2}+1\right)}{-(1/2)} \qquad \left[\because \Gamma(n+1) = n\Gamma(n)\right]$$

$$= -2\Gamma\left(\frac{1}{2}\right) \qquad \left[\because \Gamma(n) = \frac{\Gamma(n+1)}{n}\right]$$

$$= -2\sqrt{\pi}$$

(x) Let $I = \int_{0}^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx$

Put $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{1}{2\sqrt{t}} dt$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore I = \int_{0}^{\infty} \frac{e^{-t}}{\sqrt{t^{1/2}}} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_{0}^{\infty} \frac{e^{-t}}{t^{1/4} t^{1/2}} dt$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{-3}{4}} e^{-t}\,dt = \frac{1}{2}\Gamma\left(\frac{-3}{4}+1\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right) \qquad \blacksquare$$

4. *Prove the following*

(i) $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$ (ii) $\Gamma(n) = a^n \int_0^\infty e^{-ax} x^{n-1}\,dx$

(iii) $\Gamma(n) = 2a^n \int_0^\infty t^{2n-1} e^{-at^2}\,dt$ (iv) $\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-t^{\frac{1}{n}}}\,dt$

(v) $\frac{1}{n}\Gamma\left(\frac{1}{n}\right) = \int_0^\infty e^{-x^n}\,dx$ (vi) $\int_0^\infty x^n e^{-a^2x^2}\,dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$

➡ (i) RHS $= \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$

Put $\log\left(\frac{1}{x}\right) = t \Rightarrow \frac{1}{x} = e^t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt$

If $x = 0 \Rightarrow t = \infty$, if $x = 1 \Rightarrow t = 0$

$$\text{RHS} = \int_\infty^0 t^{n-1}\left(-e^{-t}dt\right) = -\int_\infty^0 t^{n-1}e^{-t}dt$$

$$= \int_0^\infty t^{n-1}e^{-t}dt = \Gamma(n) = LHS$$

$$\therefore \Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$$

(ii) Put $ax = t \Rightarrow a\,dx = dt \Rightarrow dx = \frac{dt}{a}$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty e^{-ax}x^{n-1}dx = \int_0^\infty e^{-t}\left(\frac{t}{a}\right)^{n-1}\frac{dt}{a}$$

$$= \int_0^\infty e^{-t} \frac{t^{n-1}}{a^{n-1}} . \frac{dt}{a} = \frac{1}{a^n} \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \int_0^\infty e^{-ax} x^{n-1} dx = \frac{1}{a^n} \Gamma(n)$$

$$\therefore \Gamma(n) = a^n \int_0^\infty e^{-ax} x^{n-1} dx$$

(iii) Put $at^2 = x \Rightarrow t^2 = \frac{x}{a}$

$2atdt = dx \Rightarrow tdt = \frac{dx}{2a}$

If $t = 0 \Rightarrow x = 0$, if $t = \infty \Rightarrow x = \infty$

$$\therefore \int_0^\infty t^{2n-1} e^{-at^2} dt = \int_0^\infty t^{2n-2} e^{-at^2} tdt$$

$$= \int_0^\infty (t^2)^{n-1} e^{-at^2} tdt = \int_0^\infty \left(\frac{x}{a} \right)^{n-1} e^{-x} \frac{dx}{2a}$$

$$= \int_0^\infty \frac{x^{n-1}}{a^{n-1}} e^{-x} \frac{dx}{2a} = \frac{1}{2a^n} \int_0^\infty x^{n-1} e^{-x} dx$$

$$\int_0^a t^{2n-1} e^{-at^2} dt = \frac{1}{2a^n} \Gamma(n)$$

$$\therefore \Gamma(n) = 2a^n \int_0^\infty t^{2n-1} e^{-at^2} dt$$

(iv) Put $t^{\frac{1}{n}} = x \Rightarrow t = x^n \Rightarrow dt = nx^{n-1} dx$
If $t = 0 \Rightarrow x = 0$, if $t = \infty \Rightarrow x = \infty$

$$\therefore \int_0^\infty .e^{-t^{\frac{1}{n}}} dt = \int_0^\infty e^{-x} nx^{n-1} dx$$

$$= n \int_0^\infty e^{-x} x^{n-1} dx$$

$$\int_0^\infty e^{-t^{\frac{1}{n}}} dt = n\Gamma(n)$$

$$\therefore \Gamma(n) = \frac{1}{n}\int_0^\infty e^{-t^{\frac{1}{n}}} dt$$

(v) Put $x^n = t \Rightarrow x = t^{\frac{1}{n}} \Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

If $t = 0 \Rightarrow x = 0$, if $t = \infty \Rightarrow x = \infty$

$$\therefore \int_0^\infty e^{-x^n} dx = \int_0^\infty e^{-t}\left[\frac{1}{n} t^{\frac{1}{n}-1} dt\right]$$

$$= \frac{1}{n}\int_0^\infty e^{-t} t^{\frac{1}{n}-1} dt$$

$$\int_0^\infty e^{-x^n} dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$$

$$\therefore \frac{1}{n}\Gamma\left(\frac{1}{n}\right) = \int_0^\infty e^{-x^n} dx$$

(vi) Put $a^2x^2 = t \Rightarrow x^2 = \frac{t}{a^2} \Rightarrow x = \frac{1}{a} t^{\frac{1}{2}}$

$$dx = \frac{1}{a}\frac{1}{2} t^{-\frac{1}{2}} dt = \frac{1}{2a} t^{-\frac{1}{2}} dt$$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty x^n e^{-a^2x^2} dx = \int_0^\infty \left(\frac{1}{a} t^{\frac{1}{2}}\right)^n e^{-t} \frac{1}{2a} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2a^{n+1}}\int_0^\infty t^{\frac{n}{2}-\frac{1}{2}} e^{-t} dt = \frac{1}{2a^{n+1}}\int_0^\infty t^{\frac{n-1}{2}} e^{-t} dt$$

$$= \frac{1}{2a^{n+1}}\Gamma\left(\frac{n-1}{2}+1\right) = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n-1+2}{2}\right)$$

$$\therefore \int_0^\infty x^n e^{-a^2x^2dx} = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$$ ■

5. *Prove that* $\int_0^\infty a^{-bx^2}\,dx = \dfrac{\sqrt{\pi}}{2\sqrt{b\,\log a}}$

➡ $\int_0^\infty a^{-bx^2}\,dx = \int_0^\infty e^{\log a^{-bx^2}}\,dx = \int_0^\infty e^{-bx^2\log a}\,dx$ $\qquad \because e^{\log x} = x$

Put $bx^2\log a = t \Rightarrow x^2 = \dfrac{t}{b\log a} \Rightarrow x = \dfrac{\sqrt{t}}{\sqrt{b\log a}}$

$$dx = \frac{1}{\sqrt{b\log a}}\cdot\frac{1}{2\sqrt{t}}dt = \frac{dt}{2\sqrt{t}\sqrt{b\log a}}$$

If $x = 0 \Rightarrow t = 0$, if $x = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty a^{-bx^2}\,dx = \int_0^\infty e^{-t}\frac{1}{2\sqrt{t}\sqrt{b\log a}}dt$$

$$= \frac{1}{2\sqrt{b\log a}}\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt$$

$$= \frac{1}{2\sqrt{b\log a}}\Gamma\left(\frac{-1}{2}+1\right) = \frac{1}{2\sqrt{b\log a}}\Gamma\left(\frac{1}{2}\right)$$

$$\int_0^\infty a^{-bx^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{b\log a}} \quad ■$$

6. *If n is a positive integer then prove the following.*

(i) $\quad \Gamma\left(n+\dfrac{1}{2}\right) = \dfrac{1.3.5.............(2n-1)}{2^n}\sqrt{\pi}$

(ii) $\quad \Gamma\left(-n+\dfrac{1}{2}\right) = \dfrac{(-1)^n 2^n\sqrt{\pi}}{1.3.5..........(2n-1)}$

➡ We shall prove the results by mathematical induction.

(i) (1) If $n = 1$

$$\text{LHS} = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \text{RHS}$$

Therefore, the result is true for $n = 1$

(2) Assume the result is true for $n = m$

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{1.3.5..........(2m-1)}{2^m}\sqrt{\pi} \qquad \text{---(1)}$$

(3) If $n = m+1$ the given equation becomes,

$$\Gamma\left(m+1+\frac{1}{2}\right) = \frac{1.3.5..........(2m-1)(2m+1)}{2^{m+1}}\sqrt{\pi}$$

$$\text{LHS} = \Gamma\left(m+1+\frac{1}{2}\right) = \Gamma\left(m+\frac{1}{2}+1\right)$$

$$= \left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)$$

$$= \left(\frac{2m+1}{2}\right)\left[\frac{1.3.5..........(2m-1)}{2^m}\sqrt{\pi}\right] \quad \text{[from equation (1)]}$$

$$= \frac{1.3.5..............(2m-1)(2m+1)}{2^{m+1}}\sqrt{\pi}$$

$$= \text{RHS}$$

Therefore, the result is true for $n = m+1$

Hence, the result is true for all the values of n

$$\text{i.e., } \Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5.......(2n-1)}{2^n}\sqrt{\pi}$$

(ii) (1) If $n = 1$

$$\Gamma\left(-1+\frac{1}{2}\right) = \Gamma\left(\frac{-1}{2}\right)$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right)$$

$$= -2\sqrt{\pi} \qquad \left[\because \Gamma(n) = \frac{\Gamma(n+1)}{n}\right]$$

Therefore, the result is true for $n = 1$

(2) Assume the result is true for $n = m$

$$\Gamma\left(-m+\frac{1}{2}\right) = \frac{(-1)^m 2^m \sqrt{\pi}}{1.3.5........(2m-1)}$$

(3) If $n = m + 1$ the given equation becomes,

$$\Gamma\left(-(m+1)+\frac{1}{2}\right) = \frac{(-1)^{m+1} 2^{m+1} \sqrt{\pi}}{1.3.5.........(2m-1)(2m+1)}$$

$$\text{LHS} = \Gamma\left(-(m+1)+\frac{1}{2}\right)$$

$$= \Gamma\left(-m-1+\frac{1}{2}\right) = \Gamma\left(-m-\frac{1}{2}\right)$$

$$= \Gamma\left(-\left(m+\frac{1}{2}\right)\right)$$

$$= \frac{\Gamma\left(-\left(m+\frac{1}{2}\right)+1\right)}{-\left(m+\frac{1}{2}\right)} = \frac{\Gamma\left(-m-\frac{1}{2}+1\right)}{\frac{-(2m+1)}{2}}$$

$$= \frac{-2\Gamma\left(-m+\frac{1}{2}\right)}{2m+1}$$

$$= \frac{-2}{2m+1}\left[\frac{(-1)^m 2^m \sqrt{\pi}}{1.3.5..........(2m-1)}\right]$$

$$= \frac{(-1)^{m+1} 2^{m+1} \sqrt{\pi}}{1.3.5........(2m-1)(2m+1)}$$

Therefore, the result is true for $n = m + 1$.

Hence, the result is true for all the values of n

$$\text{i.e., } \Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{1.3.5.......(2n-1)}$$ ■

Exercises

I. *Evaluate the following*

1) $\Gamma\left(\frac{5}{2}\right)$ 2) $\Gamma(4.5)$ 3) $\Gamma\left(\frac{-1}{2}\right)$ 4) $\dfrac{\Gamma\left(\frac{8}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$

5) $\dfrac{\Gamma(7)}{2\Gamma(4)\Gamma(3)}$ 6) $\dfrac{\Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$ 7) $\dfrac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{4}{3}\right)}$ 8) $\Gamma\left(\frac{-7}{2}\right)$

II. 1) $\int_0^\infty \sqrt{x}\, e^{-x} dx$ 2) $\int_0^\infty x^{\frac{3}{2}} e^{-x} dx$ 3) $\int_0^\infty x^3 e^{-2x} dx$

4) $\int_0^\infty x^6 e^{-2x} dx$ 5) $\int_0^\infty x^5 e^{-x^2} dx$ 6) $\int_0^\infty e^{-x^3} dx$

7) $\int_0^\infty e^{2ax-x^2} dx$ 8) $\int_0^1 \sqrt{\log\left(\frac{1}{x}\right)}\, dx$

9) $\int_0^\infty (\log x)^3 dx$ 10) $\int_0^1 (x \log x)^4 dx$

11) $\int_0^\infty 2^{-3x^2} dx$ 12) $\int_0^1 x^2 \log\left(\frac{1}{x}\right) dx$

13) $\int_0^\infty x^m e^{-ax^n} dx$ 14) $\int_0^\infty x^{\frac{-3}{2}} \left(1-e^{-x}\right) dx$

Answers

I. 1) $\frac{3}{4}\sqrt{\pi}$ 2) $\frac{105}{16}\sqrt{\pi}$ 3) $-2\sqrt{\pi}$

4) $\frac{10}{9}$ 5) 30 6) $\frac{16}{105}$

	7)	$\frac{4}{3}$	8)	$\frac{16\sqrt{\pi}}{105}$		
II.	1)	$\frac{\sqrt{\pi}}{2}$	2)	$\frac{3\sqrt{\pi}}{4}$	3)	$\frac{3}{8}$
	4)	$\frac{45}{8}$	5)	$\frac{105\sqrt{\pi}}{8}$	6)	$\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$
	7)	$\frac{\sqrt{\pi}\ e^{a^2}}{2}$	8)	$\frac{\sqrt{\pi}}{2}$	9)	-6
	10)	$\frac{94}{625}$	11)	$\frac{\sqrt{\pi}}{2\sqrt{3}\log 2}$	12)	$\frac{2}{27}$
	13)	$\frac{1}{na^{\frac{m+1}{n}}}\Gamma\left(\frac{m+1}{n}\right)$			14)	$2\sqrt{\pi}$

The Beta Function

The Beta function is denoted by $\beta(m,n)$ and is defined by

$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$, where m and n are positive real numbers.

Prove that $\beta(m,n)=\beta(n,m)$

Proof: We have,

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

We know that $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

Therefore, $\beta(m,n)=\int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}dx$

$$=\int_0^1 (1-x)^{m-1}(1-1+x)^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$\therefore \beta(m,n) = \beta(n,m)$$

Prove that $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Proof We have,

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Put $x = \frac{1}{1+t}$

$$1 - x = 1 - \frac{1}{1+t} = \frac{1+t-1}{1+t} = \frac{t}{1+t}$$

and $dx = \frac{-1}{(1+t)^2} dt$

If $x = 0 \Rightarrow t = \infty$, if $x = 1 \Rightarrow t = 0$

$$\therefore \beta(m,n) = \int_\infty^0 \left(\frac{1}{1+t}\right)^{m-1} \left(\frac{t}{1+t}\right)^{n-1} \left(\frac{-1}{(1+t)^2} dt\right)$$

$$= -\int_\infty^0 \frac{t^{n-1}}{(1+t)^{m-1+n-1+2}} dt = \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt$$

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

We know that $\beta(m,n) = \beta(n,m)$

$$\beta(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}} dx$$

$$\therefore \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$