3. *Solve* $x^2 \frac{dy}{dx} = 3x^2 - 2xy + 1$

➡ $x^2 \frac{dy}{dx} = 3x^2 - 2xy + 1$

$x^2 \frac{dy}{dx} + 2xy = 3x^2 + 1$

Dividing both the sides by x^2, we get

$\frac{dy}{dx} + \frac{2}{x} y = \frac{3x^2 + 1}{x^2}$, this is linear in y

$$P = \frac{2}{x} \qquad Q = \frac{3x^2 + 1}{x^2}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int (2/x) dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Therefore, the solution of the given equation is;

$$y(I.F) = \int Q(I.F) dx + c$$

$$yx^2 = \int \left(\frac{3x^2 + 1}{x^2} \right) x^2 dx + c = x^3 + x + c$$

$$yx^2 = x^3 + x + c$$ ■

4. *Solve* $x(1 - x^2) \frac{dy}{dx} + (2x^2 - 1) y = x^3$

➡ $x(1 - x^2) \frac{dy}{dx} + (2x^2 - 1) y = x^3$

Dividing both sides by $x(1 - x^2)$, we get

$$\frac{dy}{dx} + \frac{2x^2 - 1}{x(1 - x^2)} y = \frac{x^2}{(1 - x^2)}$$

$$P = \frac{2x^2 - 1}{x(1 - x^2)} \qquad Q = \frac{x^2}{1 - x^2}$$

$$P = \frac{2x^2 - 1}{x(1 - x^2)} = \frac{2x^2 - 1}{x(1 - x)(1 + x)} = \frac{A}{x} + \frac{B}{1 - x} + \frac{C}{1 + x}$$

$$\Rightarrow 2x^2 - 1 = A(1 - x)(1 + x) + Bx(1 + x) + Cx(1 - x)$$

Put $x = 0 \Rightarrow -1 = A$
Put $x = 1 \Rightarrow 1 = 2B \Rightarrow B = 1/2$
Put $x = -1 \Rightarrow 1 = -2C \Rightarrow C = -1/2$

$$\therefore P = \frac{-1}{x} + \frac{1/2}{1-x} - \frac{1/2}{1+x}$$

$$\int Pdx = \int\left(-\frac{1}{x} + \frac{1}{2}\frac{1}{(1-x)} - \frac{1}{2}\frac{1}{(1+x)}\right)dx$$

$$= -\log x + \frac{1}{2}\frac{\log(1-x)}{-1} - \frac{1}{2}\log(1+x)$$

$$= -\frac{1}{2}\left[2\log x + \log(1-x) + \log(1+x)\right]$$

$$= -\frac{1}{2}\log\left[x^2(1-x)(1+x)\right]$$

$$= -\frac{1}{2}\log\left[x^2(1-x^2)\right]$$

$$\int Pdx = \log\left[x^2(1-x^2)\right]^{-\frac{1}{2}}$$

$$\text{I.F} = e^{\int Pdx} = e^{\log\left[x^2(1-x^2)\right]^{-\frac{1}{2}}}$$

$$\text{I.F} = \left[x^2(1-x^2)\right]^{-\frac{1}{2}} = \frac{1}{x\sqrt{1-x^2}}$$

Therefore, the solution of the given equation is

$$y(I.F) = \int Q(I.F)dx + c$$

$$y\left(\frac{1}{x\sqrt{1-x^2}}\right) = \int \frac{x^2}{1-x^2}\frac{1}{x\sqrt{1-x^2}}dx + c$$

$$= \int \frac{x}{(1-x^2)^{\frac{3}{2}}}dx + c$$

Put $1 - x^2 = t \Rightarrow -2xdx = dt \Rightarrow xdx = -\frac{dt}{2}$

$$\frac{y}{x\sqrt{1-x^2}} = \int \frac{-dt/2}{t^{3/2}} + c$$

$$= -\frac{1}{2}\int t^{-\frac{3}{2}} dt + c = -\frac{1}{2}\frac{t^{-\frac{1}{2}}}{(-1/2)} + c = \frac{1}{\sqrt{t}} + c$$

$$\frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + c$$

$$\therefore\ y = x + cx\sqrt{1-x^2}$$ ■

5. *Solve* $dx + xdy = e^{-y}\sec y dy$

➡ $dx + xdy = e^{-y}\sec y dy$

$\frac{dx}{dy} + x = e^{-y}\sec y$, this is linear in x

$P = 1,\ Q = e^{-y}\sec y$

$I.F = e^{\int P dy} = e^{\int 1 dy} = e^{y}$

Therefore, the solution of the given equation is

$$x(I.F.) = \int Q(I.F.)dy + c$$

$$xe^{y} = \int e^{-y}\sec y e^{y} dy + c = \int \sec y dy + c$$

$$xe^{y} = \log(\sec y + \tan y) + c$$ ■

6. *Solve* $ye^{y}dx = \left(y^3 + 2xe^{y}\right)dy$

➡ $ye^{y}dx = \left(y^3 + 2xe^{y}\right)dy$, this equation can be written as

$$\frac{dx}{dy} = \frac{y^3 + 2xe^{y}}{ye^{y}}$$

$$\frac{dx}{dy} = \frac{y^2}{e^{y}} + \frac{2x}{y}$$

$\frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}$, this is linear in x

$P = \frac{-2}{y},\ Q = y^2 e^{-y}$

$$\therefore I.F = e^{\int P dy} = e^{\int -\frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2}$$

Therefore, the solution of the given equation is

$$x(I.F) = \int Q(I.F)\,dy + c$$

$$xy^{-2} = \int y^2 e^{-y} y^{-2} dy + c = \frac{e^{-y}}{-1} + c$$

$$\frac{x}{y^2} = -e^{-y} + c$$

$$\frac{x}{y^2} + e^{-y} = c$$

7. *Solve* $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

➡ $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

$$\frac{dy}{dx} + 2x\sin y.\cos y = x^3 \cos^2 y$$

Dividing on both the sides by $\cos^2 y$, we get

$$\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$$

Put $\tan y = t \Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \frac{dt}{dx} + 2xt = x^3$, this is linear in t

$$P = 2x\ ,\ Q = x^3$$

$$I.F = e^{\int 2x dx} = e^{x^2}$$

Therefore, the solution of the given equation is

$$t(I.F) = \int Q(I.F)\,dx + c$$

$$\tan y\left(e^{x^2}\right) = \int x^3 e^{x^2} dx + c$$

Put $x^2 = z \Rightarrow 2xdx = dz \Rightarrow xdx = dz/2$

$$e^{x^2}\tan y = \int x^2 e^{x^2} x\,dx = \int z e^z \frac{dz}{2}$$

$$= \frac{1}{2}\left(ze^z - \int 1.e^z dz\right) + c$$

$$= \frac{1}{2}\left(ze^z - e^z\right) + c$$

$$e^{x^2}\tan y = \frac{1}{2}(x^2 e^{x^2} - e^{x^2}) + c$$ ■

8. *Solve* $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$

➡ $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right) = e^x e^{-y}\left(e^x - e^y\right)$

$$\frac{dy}{dx} = e^{2x}e^{-y} - e^x$$

$$e^y\frac{dy}{dx} = e^{2x} - e^x \cdot e^y$$

$$e^y\frac{dy}{dx} + e^x e^y = e^{2x}$$

Put $e^y = z \Rightarrow e^y \frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dz}{dx} + e^x z = e^{2x}$, this is linear in z.

$P = e^x \qquad Q = e^{2x}$

I.F $= e^{\int p\,dx} = e^{\int e^x dx} = e^{e^x}$

Therefore, the solution of the given differential equation is,

$$z(I.F) = \int Q(I.F)dx + c$$

$$e^y e^{e^x} = \int e^{2x} e^{e^x} dx + c$$

$$= \int e^x e^{e^x} e^x dx + c$$

Put $e^x = t \Rightarrow e^x dx + dt$

$$e^y e^{e^x} = \int t e^t dt + c$$

$$= te^t - \int 1 \cdot e^t dt + c$$

$$= te^t - e^t + c$$

$$= (t-1)e^t + c$$

$$e^y e^{e^x} = (e^x - 1)e^{e^x} + c$$ ■

9. *Solve* $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

➡ $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Dividing $\sec y$ on both sides, we get

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{1+x}.\frac{1}{\sec y} = (1+x)e^x$$

$$\cos y\frac{dy}{dx} - \frac{1}{1+x}\frac{\sin y}{\cos y}.\cos y = (1+x)e^x$$

$$\cos y\frac{dy}{dx} - \frac{1}{1+x}\sin y = (1+x)e^x \quad \text{---(1)}$$

Put $\sin y = t \Rightarrow \cos y\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ (1) becomes, $\frac{dt}{dx} - \frac{1}{1+x}t = (1+x)e^x$, this is linear in t

Here $P = -\frac{1}{1+x}$, $Q = (1+x)e^x$

$\therefore$ I.F. $= e^{\int pdx} = e^{\int -\frac{1}{1+x}dx} = e^{-\log(1+x)} = e^{\log(1+x)^{-1}} = (1+x)^{-1} = \frac{1}{(1+x)}$

Therefore, the solution of the given equation is

$$t(I.F) = \int Q(I.F)dx + c$$

$$\sin y\left(\frac{1}{1+x}\right) = \int (1+x)e^x\left(\frac{1}{1+x}\right)dx + c = \int e^x dx + c$$

$$\frac{\sin y}{1+x} = e^x + c$$ ■

10. *Solve* $dy+\left(2x\tan^{-1}y-x^{3}\right)\left(1+y^{2}\right)dx=0$

➡ $dy+\left(2x\tan^{-1}y-x^{3}\right)\left(1+y^{2}\right)dx=0$

$$dy=-\left(2x\tan^{-1}y-x^{3}\right)\left(1+y^{2}\right)dx$$

$$\frac{1}{1+y^{2}}\frac{dy}{dx}=-2x\tan^{-1}y+x^{3}$$

$$\frac{1}{1+y^{2}}\frac{dy}{dx}+2x\tan^{-1}y=x^{3}$$

Put $\tan^{-1}y=t\Rightarrow\frac{1}{1+y^{2}}\frac{dy}{dx}=\frac{dt}{dx}$

$\therefore\frac{dt}{dx}+2xt=x^{3}$, this is linear in t

$P=2x,\ \ Q=x^{3}$

$\text{I.F}=e^{\int 2x\,dx}=e^{x^{2}}$

Therefore, the solution of the given equation is

$$t(I.F)=\int Q(I.F)\,dx+c$$

$$\tan^{-1}y\left(e^{x^{2}}\right)=\int x^{3}e^{x^{2}}dx+c$$

Put $x^{2}=z\Rightarrow 2xdx=dz\Rightarrow xdx=\frac{dz}{2}$

$$e^{x^{2}}\tan^{-1}y=\int x^{2}e^{x^{2}}\,xdx+c$$

$$=\int ze^{z}\frac{dz}{2}+c=\frac{1}{2}(ze^{z}-\int 1.e^{z}dz)+c$$

$$=\frac{1}{2}(ze^{z}-e^{z})+c$$

$$e^{x^{2}}\tan^{-1}y=\frac{1}{2}(x^{2}e^{x^{2}}-e^{x^{2}})+c$$ ■

11. *Solve* $\left(\sin^{3}x\right)y'+y\sin x=\cos x$

➡ $\left(\sin^{3}x\right)y'+y\sin x=\cos x$

Dividing both sides by $\sin^{3}x$, we get

$$\frac{dy}{dx} + y\frac{\sin x}{\sin^3 x} = \frac{\cos x}{\sin^3 x}$$

$$\frac{dy}{dx} + y\,\text{cosec}^2 x = \cot x\,\text{cosec}^2 x$$, this is linear in y

$$P = \text{cosec}^2 x,\ Q = \cot x\,\text{cosec}^2 x$$

$$\text{I.F} = e^{\int pdx} = e^{\int \text{cosec}^2 x\,dx} = e^{-\cot x}$$

Therefore, the solution of the given differential equation is

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$ye^{-\cot x} = \int \cot x\,\text{cosec}^2 x\,e^{-\cot x}\,dx + c$$

Put $\cot x = t \Rightarrow -\text{cosec}^2 x\,dx = dt \Rightarrow \text{cosec}^2 x\,dx = -dt$

$$ye^{-\cot x} = \int te^{-t}(-dt) + c$$

$$= -\left[t\frac{e^{-t}}{-1} - \int 1.\frac{e^{-t}}{-1}\,dt\right] + c$$

$$= te^{-t} - \frac{e^{-t}}{-1} + c = te^{-t} + e^{-t} + c$$

$$= (t+1)e^{-t} + c$$

$$ye^{-\cot x} = (\cot x + 1)e^{-\cot x} + c$$

12. *Solve* $\dfrac{dy}{dx} = \dfrac{1}{e^{-y}\sec^2 y - x}$

➨ $$\frac{dy}{dx} = \frac{1}{e^{-y}\sec^2 y - x}$$

$$\Rightarrow \frac{dx}{dy} = e^{-y}\sec^2 y - x$$

$$\frac{dx}{dy} + x = e^{-y}\sec^2 y$$, This is linear in x

$$P = 1, \quad Q = e^{-y}\sec^2 y$$

$$I.F = e^{\int Pdy} = e^{\int dy} = e^{y}$$

Therefore, the solution of the given equation is

$$x(I.F) = \int Q(I.F)\,dy + c$$
$$xe^{y} = \int e^{-y}\sec^2 y\, e^{y}\,dy + c$$
$$xe^{y} = \tan y + c$$ ■

13. *Solve* $\left(1+x^2\right)\dfrac{dy}{dx} + 2xy - 6x^2 = 0$

➡ $\left(1+x^2\right)\dfrac{dy}{dx} + 2xy - 6x^2 = 0$

$$\left(1+x^2\right)\frac{dy}{dx} + 2xy = 6x^2$$

Dividing both the sides by $1+x^2$, we get

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{6x^2}{1+x^2}$$, this is linear in y

$$P = \frac{2x}{1+x^2} \qquad Q = \frac{6x^2}{1+x^2}$$

$$I.F = e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log\left(1+x^2\right)} = \left(1+x^2\right)$$

Therefore, the solution of the given equation is

$$y(I.F) = \int Q(I.F)\,dx + c$$
$$y\left(1+x^2\right) = \int \frac{6x^2}{1+x^2}\left(1+x^2\right)dx + c = 6\frac{x^3}{3} + c$$
$$y\left(1+x^2\right) = 2x^3 + c$$ ■

14. *Solve* $y\cos y\, x' + x(y\sin y + \cos y) = 1$

➡ $y\cos y\, x' + x(y\sin y + \cos y) = 1$

$$y\cos y\frac{dx}{dy} + (y\sin y + \cos y)x = 1$$

Dividing both the sides by $y\cos y$, we get

$$\frac{dx}{dy} + \frac{y\sin y + \cos y}{y\cos y}x = \frac{1}{y\cos y}$$, this is linear in x

$$P = \frac{y\sin y + \cos y}{y\cos y}, \qquad Q = \frac{1}{y\cos y}$$

$$\text{I.F} = e^{\int \frac{y\sin y+\cos y}{y\cos y}dy} = e^{\int\left(\tan y+\frac{1}{y}\right)dy} = e^{(\log(\sec y)+\log y)} = e^{\log(y\sec y)} = y\sec y$$

Therefore, the solution of the given equation is

$$x(I.F) = \int Q(I.F)\,dy + c$$

$$xy\sec y = \int \frac{1}{y\cos y}(y\sec y)\,dy + c = \int \sec^2 y\,dy + c$$

$$xy\sec y = \tan y + c$$ ■

15. *Solve* $\frac{dy}{dx}x\cos x + y(\cos x - x\sin x) = 1$

➡ $\frac{dy}{dx}x\cos x + y(\cos x - x\sin x) = 1,$

Dividing both the sides by $x\cos x$, we get

$$\frac{dy}{dx} + y\left(\frac{\cos x - x\sin x}{x\cos x}\right) = \frac{1}{x\cos x}$$, this is linear in y

$$P = \frac{\cos x - x\sin x}{x\cos x} = \frac{1}{x} - \tan x, \qquad Q = \frac{1}{x\cos x}$$

$$I.F = e^{\int P dx} = e^{\int\left(\frac{1}{x}-\tan x\right)dx} = e^{(\log x - \log\sec x)} = e^{\log\left(\frac{x}{\sec x}\right)} = \frac{x}{\sec x}$$

Therefore, the solution of the given equation is

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$y\frac{x}{\sec x} = \int \frac{1}{x\cos x}\left(\frac{x}{\sec x}\right)dx + c = \int dx + c$$

$$xy\cos x = x + c$$ ■

16. *Solve* $2y'\cos x + 4y\sin x = \sin 2x$ *given* $y = 0$ *when* $x = \pi/3$

➡ $2y'\cos x + 4y\sin x = \sin 2x,$

$$2\frac{dy}{dx}\cos x + 4y\sin x = 2\sin x\cos x$$