Differentiating again with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = m(m-2)r^{m-3}\frac{\partial r}{\partial x}x + mr^{m-2} = m(m-2)\,r^{m-3}\cdot x\cdot\frac{x}{r} + mr^{m-2}$$

$$\frac{\partial^2 u}{\partial x^2} = m(m-2)r^{m-4}.x^2 + mr^{m-2}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = m(m-2)r^{m-4}.y^2 + mr^{m-2}$ and

$$\frac{\partial^2 u}{\partial z^2} = m(m-2)r^{m-4}z^2 + mr^{m-2}$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m-2)r^{m-4}\left[x^2+y^2+z^2\right] + 3mr^{m-2}$$

$$= m(m-2)r^{m-4}\left[r^2\right] + 3mr^{m-2}$$

$$= m(m-2)r^{m-2} + 3mr^{m-2} = r^{m-2}\left[m^2-2m+3m\right]$$

$$= r^{m-2}\left[m(m+1)\right]$$ ■

12. *If* $u = f(x+ay) + g(x-ay)$, *prove that* $\frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}$

➡ $u = f(x+ay) + g(x-ay)$ ---(1)

Differentiating (1) partially with respect to x, we get

$\frac{\partial u}{\partial x} = f'(x+ay)\cdot 1 + g'(x-ay)\cdot 1$, differentiating again, we get

$$\frac{\partial^2 u}{\partial x^2} = f''(x+ay)\cdot 1 + g''(x-ay)\cdot 1 \qquad \text{---(2)}$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = f'(x+ay).a + g'(x-ay)(-a)$$

$\frac{\partial u}{\partial y} = af'(x+ay) - ag'(x-ay)$, differentiating again, we get

$$\frac{\partial^2 u}{\partial y^2} = af''(x+ay)a - ag''(x-ay)(-a)$$

$$= a^2 f''(x+ay) + a^2 g''(x-ay)$$

$$= a^2\left[f''(x+ay) + g''(x-ay)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

■

13. *If* $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$*, show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

➡ $u = \frac{1}{\sqrt{x^2+y^2+z^2}} = \left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$

Differentiating with respect to x, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2x) = -x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

Differentiating again with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = -\left[x\left(\frac{-3}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}(2x) + 1\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]$$

$$= -\left[-3x^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}} + \left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 3x^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}} - \left(x^2+y^2+z^2\right)^{-\frac{3}{2}} \quad \text{---(1)}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = 3y^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}} - \left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$ ---(2)

$$\frac{\partial^2 u}{\partial z^2} = 3z^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}} - \left(x^2+y^2+z^2\right)^{-\frac{3}{2}} \quad \text{---(3)}$$

Adding equations (1), (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$=3\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}-3\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$=3\left(x^2+y^2+z^2\right)^{-\frac{5}{2}+1}-3\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$=3\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}-3\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}=0 \quad ■$$

14. *If* $u=f(r)$ *where* $r=\sqrt{x^2+y^2+z^2}$ *then prove that*

$$u_{xx}+u_{yy}+u_{zz}=f''(r)+\frac{2}{r}f'(r)$$

➡ $r=\sqrt{x^2+y^2+z^2}$

$$\Rightarrow r^2=x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$$

Similarly,

$$\frac{\partial r}{\partial y}=\frac{y}{r}$$

$$\frac{\partial r}{\partial z}=\frac{z}{r}$$

Given $u=f(r)$

$$\frac{\partial u}{\partial x}=f'(r)\frac{\partial r}{\partial x}=\frac{x}{r}f'(r)=\frac{xf'(r)}{r}$$

$$u_{xx}=\frac{\partial^2 u}{\partial x^2}=\frac{1}{r^2}\left[r\frac{\partial}{\partial x}(xf'(r))-xf'(r)\frac{\partial r}{\partial x}\right]$$

$$\therefore\frac{\partial^2 u}{\partial x^2}=\frac{1}{r^2}\left[r\left[xf''(r)\frac{\partial r}{\partial x}+f'(r)\right]-xf'(r)\frac{\partial r}{\partial x}\right]$$

$$=\frac{1}{r^2}\left[rxf''(r)\frac{x}{r}+rf'(r)-xf'(r)\cdot\frac{x}{r}\right]$$

$$= \frac{1}{r^2}\left[x^2 f''(r) + rf'(r) - \frac{x^2 f'(r)}{r}\right]$$

$$u_{xx} = \frac{1}{r^2}\left[x^2 f''(r) + f'(r)\left\{r - \frac{x^2}{r}\right\}\right] \quad \text{---(1)}$$

Similarly,

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[y^2 f''(r)\left\{r - \frac{y^2}{r}\right\}\right] \quad \text{---(2)}$$

$$u_{zz} = \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}\left[z^2 f''(r) + f'(r)\left\{r - \frac{z^2}{r}\right\}\right] \quad \text{---(3)}$$

Adding equations (1), (2) and (3), we get

$$u_{xx} + u_{yy} + u_{zz} = \frac{1}{r^2}\left[x^2 f''(r) + f'(r)\left\{r - \frac{x^2}{r}\right\}\right] + \frac{1}{r^2}\left[y^2 f''(r) + f'(r)\left\{r - \frac{y^2}{r}\right\}\right] + \frac{1}{r^2}\left[z^2 f''(r) + f'(r)\left\{r - \frac{z^2}{r}\right\}\right]$$

$$= \frac{1}{r^2}\left[f''(r)\left(x^2 + y^2 + z^2\right) + f'(r)\left\{r - \frac{x^2}{r} + r - \frac{y^2}{r} + r - \frac{z^2}{r}\right\}\right]$$

$$= \frac{1}{r^2}\left[f''(r)\left(r^2\right) + f'(r)\left\{3r - \frac{r^2}{r}\right\}\right]$$

$$= \frac{1}{r^2}\left[f''(r)r^2 + f'(r)(2r)\right]$$

$$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$$ ■

15. *If* $x = r\cos\theta,\ y = r\sin\theta$, *show that*

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

➡ We have,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$x^2 + y^2 = r^2$$

Differentiating with respect to x, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating again with respect to x, we get

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r^2}\left[r\cdot 1 - x\frac{\partial r}{\partial x}\right] = \frac{1}{r^2}\left[r - x\left(\frac{x}{r}\right)\right] = \frac{1}{r^2}\left[r - \frac{x^2}{r}\right]$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r^3}\left[r^2 - x^2\right]$$

Similarly, $\dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r^3}\left[r^2 - y^2\right]$

Therefore, $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{r^2 - x^2}{r^3} + \dfrac{r^2 - y^2}{r^3}$

$$= \frac{r^2 - x^2 + r^2 - y^2}{r^3} = \frac{2r^2 - \left(x^2 + y^2\right)}{r^3}$$

$$= \frac{2r^2 - r^2}{r^3} = \frac{r^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = \frac{1}{r} \qquad \text{---(1)}$$

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1 \qquad \text{---(2)}$$

From equations (1) and (2), we get

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$ ■

Exercises

1. *Find* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *when,*

   (i) $f = \log(x^2 + y^2)$ (ii) $f = x^y$

   (iii) $f = \sin^{-1}(y/z)$

2. *Prove the relation* $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ *for the following functions,*

   (i) $f = ax^2 - 2hxy + by^2$ (ii) $f = x \log x$

   (iii) $f = x \sin y + y \sin x$ (iv) $f = x^3 + y^3 - 3axy$

   (v) $f = \sin^{-1}\left(\frac{x}{y}\right)$ (vi) $f = \log(x \sin y + y \sin x)$

3. *If* $u = e^{xyz}$, *show that* $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2) e^{xyz}$

4. *If* $f = f\left(\frac{y}{x}\right)$, *prove that* $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0$

5. *If* $x^x y^y z^z = c$, *show that at*

   $x = y = z, \frac{\partial^2 z}{\partial x \partial y} = -(x \log_e x)^{-1}$

6. *If* $u = \frac{x^2 + y^2}{\sqrt{x + y}}$, *show that* $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u.$

7. *If* $u = e^{(x^3 + y^3)}$, *prove that* $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u \log u$

8. *If* $u = \frac{xy}{x + y}$, *show that*

   (i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$, (ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$

9. *If* $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, *prove that* $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

10. *If* $u = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$, *show that* $\dfrac{\partial^2 u}{\partial x^2} = a^2 \dfrac{\partial^2 u}{\partial y^2}$

11. *If* $z = \cos(x + y) + \sin(x - y)$, *prove that* $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$

12. *If* $z = \dfrac{x^2 + y^2}{x + y}$, *prove that* $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left[\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right]$

13. *If* $u = \log(\tan x + \tan y + \tan z)$, *prove that*

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

14. *If* $z = f(y - 3x) + y(y + 2x) + \sin x - y\cos x$, *prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 6\frac{\partial^2 u}{\partial y^2} = y\cos x$$

15. *If* $u = r^n$ *where* $r^2 = x^2 + y^2 + z^2$, *prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = n(n+1)r^{n-2}$$

16. If $u = (1 - 2x + y^2)^{-\frac{1}{2}}$, *prove that* $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = y^2 u^3$

Answers

1. (i) $\dfrac{\partial f}{\partial x} = \dfrac{\partial x}{x^2 + y^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2 + y^2}$

   (ii) $\dfrac{\partial f}{\partial x} = x^y \dfrac{y}{x}$, $\dfrac{\partial f}{\partial y} = x^y \log x$

   (iii) $\dfrac{\partial f}{\partial x} = -\dfrac{y}{x\sqrt{x^2 - y^2}}$, $\dfrac{\partial f}{\partial y} = \dfrac{1}{\sqrt{x^2 - y^2}}$

HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ of two independent variables x and y is called *homogeneous function of degree n* if the function can be written in the form $x^n g(y/x)$ or $y^n h(x/y)$.

Illustrations

(a) $f(x, y) = x^2 + xy + 2y^2 = x^2\left(1 + \frac{y}{x} + 2\left(\frac{y}{x}\right)^2\right) = x^2 g\left(\frac{y}{x}\right)$

is a homogeneous function of degree 2.

(b) $f(x, y) = \frac{x^4 - y^4}{x + y} = \frac{x^4\left(1 - (y/x)^4\right)}{x\left(1 + (y/x)\right)} = x^3 g\left(\frac{y}{x}\right)$

is a homogeneous of degree 3.

(c) $f(x, y) = \cos(y/x) = x^0 \cos(y/x) = x^0 g(y/x)$

is a homogeneous function of degree 0.

Euler's Theorem of Homogeneous Functions

If $f(x, y)$ be a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof If $f(x, y)$ is homogenous function of degree n, then

$$f(x, y) = f = x^n g\left(\frac{y}{x}\right)$$

Differentiating, f with respect to x, we get

$$\frac{\partial f}{\partial x} = x^n g\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + nx^{n-1} g\left(\frac{y}{x}\right)$$

$$x\frac{\partial f}{\partial x} = -yx^{n-1} g\left(\frac{y}{x}\right) + nx^n g\left(\frac{y}{x}\right) \qquad \text{---(1)}$$

Next, differentiating f with respect y, we get

$$\frac{\partial f}{\partial y} = x^n g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1} g'\left(\frac{y}{x}\right)$$

Therefore, $y\frac{\partial f}{\partial y} = yx^{n-1}g'\left(\frac{y}{x}\right)$ ---(2)

Adding equations (1) and (2), we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -yx^{n-1}g'\left(\frac{y}{x}\right) + nx^n g\left(\frac{y}{x}\right) + yx^{n-1}g'\left(\frac{y}{x}\right)$$

$$= nx^n g(y/x) = nf$$

Hence the proof.

In general if $f(x_1, x_2, \ldots\ldots x_m)$ is a homogenous function of degree n, then

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} + \ldots\ldots\ldots + x_m\frac{\partial f}{\partial x_m} = nf$$

Euler's Extension Theorem

If f is a homogeneous function of x and y with degree n then,

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Proof By using Euler's theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \qquad \text{---(1)}$$

Differentiating equation (1) partially with respect to x, we get

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x\partial y} = n\frac{\partial f}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x\partial y} = (n-1)\frac{\partial f}{\partial x}$$, multiply x on both sides,

$$x^2\frac{\partial^2 f}{\partial x^2} + xy\frac{\partial^2 f}{\partial x\partial y} = x(n-1)\frac{\partial f}{\partial x} \qquad \text{---(2)}$$

Differentiate (1) partially with respect to y, we get

$$x\frac{\partial^2 f}{\partial x\partial y} + y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = n\frac{\partial f}{\partial y}$$

$$\Rightarrow y\frac{\partial^2 f}{\partial y^2} + x\frac{\partial^2 f}{\partial x\partial y} = (n-1)\frac{\partial f}{\partial y}$$, multiply y on both sides

$$y^2\frac{\partial^2 f}{\partial y^2}+xy\frac{\partial^2 f}{\partial x\partial y}=y(n-1)\frac{\partial f}{\partial y} \qquad \text{---(3)}$$

Adding equations (2) and (3), we get

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=(n-1)\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]$$

$$=(n-1)nf$$

Worked Examples

1. *Verify the Euler's theorem for* $u=\dfrac{xy}{x+y}$

➡ $u=\dfrac{xy}{x+y}=x\left(\dfrac{y/x}{1+y/x}\right)=xg\left(\dfrac{y}{x}\right)$

Therefore, the given function is homogeneous with degree 1. hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.u$$

$$u=\frac{xy}{x+y}$$

$$\frac{\partial u}{\partial x}=\frac{(x+y)y-xy(1)}{(x+y)^2}=\frac{xy+y^2-xy}{(x+y)^2}=\frac{y^2}{(x+y)^2}$$

$$\frac{\partial u}{\partial y}=\frac{(x+y)x-xy(1)}{(x+y)^2}=\frac{x^2}{(x+y)^2}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{xy^2}{(x+y)^2}+\frac{yx^2}{(x+y)^2}=\frac{xy^2+yx^2}{(x+y)^2}=\frac{xy(y+x)}{(x+y)^2}$$

$$=\frac{xy}{(x+y)}=u$$

$\therefore x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=u$ Hence verified. ■

2. *Verify Euler's theorem for the function* $u = x\tan^{-1}(y/x)$

➡ $u = x\tan^{-1}(y/x)$

Here u is homogeneous function with degree 1, hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u$$

$$u = x\tan^{-1}(y/x)$$

$$\frac{\partial u}{\partial x} = x\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) + 1.\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{xy}{x^2+y^2} + \tan^{-1}\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} = -\frac{x^2 y}{x^2+y^2} + x\tan^{-1}\left(\frac{y}{x}\right)$$

and $$\frac{\partial u}{\partial y} = x\frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{x^2 y}{x^2+y^2}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{x^2 y}{x^2+y^2} + x\tan^{-1}\left(\frac{y}{x}\right) + \frac{x^2 y}{x^2+y^2}$$

$$= x\tan^{-1}\left(\frac{y}{x}\right) = u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$ Hence verified. ■

3. *Verify Euler's theorem for the function*

$$u = \left(x^{1/2} + y^{1/2}\right)\left(x^n + y^n\right)$$

➡ $u = \left(x^{1/2} + y^{1/2}\right)\left(x^n + y^n\right)$

$$= x^{1/2}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right) x^n\left(1 + \frac{y^n}{x^n}\right)$$

$$= x^{1/2} \cdot x^n \left(1+\left(\frac{y}{x}\right)^{1/2}\right)\left(1+\left(\frac{y}{x}\right)^n\right)$$

$$= x^{n+1/2} g\left(\frac{y}{x}\right)$$

Therefore, the given function is a homogeneous with degree $n+\frac{1}{2}$.

Hence, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n+\frac{1}{2}\right)u$$

$$u = \left(x^{1/2} + y^{1/2}\right)\left(x^n + y^n\right)$$

$$\frac{\partial u}{\partial x} = \left(x^{1/2} + y^{1/2}\right)\left[nx^{n-1}\right] + \left(x^n + y^n\right)\frac{1}{2}x^{-1/2}$$

$$x\frac{\partial u}{\partial x} = \left(x^{1/2} + y^{1/2}\right)nx^n + \left(x^n + y^n\right)\frac{1}{2}x^{1/2} \qquad \text{---(1)}$$

$$\frac{\partial u}{\partial y} = \left(x^{1/2} + y^{1/2}\right)ny^{n-1} + \left(x^n + y^n\right)\frac{1}{2}y^{-1/2}$$

$$y\frac{\partial u}{\partial y} = \left(x^{1/2} + y^{1/2}\right)ny^n + \left(x^n + y^n\right)\frac{1}{2}y^{1/2} \qquad \text{---(2)}$$

Adding equations (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(x^{1/2} + y^{1/2}\right)nx^n + \left(x^n + y^n\right)\frac{1}{2}x^{1/2}$$

$$+\left(x^{1/2} + y^{1/2}\right)ny^n + \left(x^n + y^n\right)\frac{1}{2}y^{1/2}$$

$$= n\left(x^{1/2} + y^{1/2}\right)\left(x^n + y^n\right) + \frac{1}{2}\left(x^n + y^n\right)\left(x^{1/2} + y^{1/2}\right)$$

$$= nu + \frac{1}{2}u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n+\frac{1}{2}\right)u$$

■

4. *If* $u = e^{x^3+y^3}$, *then prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u\log u$

➡ $u = e^{x^3+y^3}$

$$\log u = x^3 + y^3 = x^3\left(1 + \frac{y^3}{x^3}\right)$$

$$= x^3\left(1 + \left(\frac{y}{x}\right)^3\right) = x^3 h\left(\frac{y}{x}\right) = f\,(say)$$

Therefore, f is homogeneous function of degree 3. Hence, by Euler's theorem we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$$

$$x\frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) = 3\log u$$

$$x\cdot\frac{1}{u}\frac{\partial u}{\partial x} + y\cdot\frac{1}{u}\frac{\partial u}{\partial y} = 3\log u$$

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 3\log u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u \qquad \blacksquare$$

5. *If* $u = \dfrac{xy^3 + x^3 y\sin(y/x)}{x^7 y + x^8 e^{y/x}}$, *then prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -4u.$

➡ $u = \dfrac{xy^3 + x^3 y\sin(y/x)}{x^7 y + x^8 e^{y/x}} = \dfrac{x^4\left[(y/x)^3 + (y/x)\sin(y/x)\right]}{x^8\left[(y/x) + e^{y/x}\right]}$

$$= \frac{x^{-4}\left[(y/x)^3 + (y/x)\sin(y/x)\right]}{\left[(y/x) + e^{y/x}\right]} = x^{-4} g(y/x)$$

Therefore, u is homogeneous function with degree – 4, hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-4u \qquad \blacksquare$$

6. *If* $u=\sin^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ *prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u$

➡ $u=\sin^{-1}\left[\frac{x^3+y^3}{x+y}\right]$

$$\sin u=\frac{x^3+y^3}{x+y}=\frac{x^3\left(1+(y/x)^3\right)}{x[1+(y/x)]}=x^2\left[\frac{1+(y/x)^3}{1+(y/x)}\right]=x^2g\left(\frac{y}{x}\right)=f$$

Now f is homogeneous function with degree 2, by using Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2f$$

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial x}(\sin u)=2\sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=2\sin u$$

$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\cos u=2\sin u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u \qquad \blacksquare$$

7. *If* $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, *prove that*

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(1-4\sin^2 u\right)\sin 2u$

➡ $\tan u=\frac{x^3\left(1+(y/x)^3\right)}{x\left(1-(y/x)\right)}=\frac{x^2\left(1+(y/x)^3\right)}{\left(1-(y/x)\right)}=x^2g\left(y/x\right)=f\left(say\right)$

Now f is homogeneous function with degree 2, hence using Euler's theorem, we have

(i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \times \cos^2 u = 2\frac{\sin u}{\cos u}\cdot\cos^2 u = 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \qquad \text{---(1)}$$

(ii) Differentiating (1) partially with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u - 1)\frac{\partial u}{\partial x}$, multiply x on both sides, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = x(2\cos 2u - 1)\frac{\partial u}{\partial x} \qquad \text{---(2)}$$

Differentiating (1) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = (2\cos 2u)\frac{\partial u}{\partial y}$$

$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\frac{\partial u}{\partial y}$, multiply y on both sides, we get

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)y\frac{\partial u}{\partial y} \qquad \text{---(3)}$$

Adding equations (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=(2\cos 2u-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=\left[2\left(1-2\sin^2 u\right)-1\right]\left[\sin 2u\right]=\left[2-4\sin^2 u-1\right]\ \left[\sin 2u\right]$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\left(1-4\sin^2 u\right)(\sin 2u)$$ ■

8. *If* $f(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$, *prove that*

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f=0$$

➡ $$f(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log(x/y)}{x^2+y^2}$$

$$=\frac{1}{y^2(x/y)^2}+\frac{1}{y^2(x/y)}+\frac{\log(x/y)}{y^2\left[(x/y)^2+1\right]}$$

$$=\frac{1}{y^2}\left[\frac{1}{(x/y)^2}+\frac{1}{x/y}+\frac{\log(x/y)}{(x/y)^2+1}\right]=y^{-2}g\left(\frac{x}{y}\right)$$

Therefore, $f(x,y)$ is homogeneous function with degree –2, hence by using Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-2f$$

i.e. $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f=0$$ ■

9. *If* $u=\operatorname{cosec}^{-1}\left(\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}\right)$, *prove that*

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{6}\tan u$ and

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{6}\left(1+\frac{1}{6}\sec^2 u\right)\tan u$

➡ $$u=\operatorname{cosec}^{-1}\left(\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}\right)$$

$$\operatorname{cosec} u=\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}=\frac{x^{\frac{1}{2}}\left(1+(y/x)^{\frac{1}{2}}\right)}{x^{\frac{1}{3}}\left(1+(y/x)^{\frac{1}{3}}\right)}$$

$$\operatorname{cosec} u=x^{\frac{1}{6}}\left(\frac{1+(y/x)^{\frac{1}{2}}}{1+(y/x)^{\frac{1}{3}}}\right)=x^{\frac{1}{6}}g(y/x)=f$$

Therefore, by Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{6}f$$

(i) $x\frac{\partial}{\partial x}(\operatorname{cosec} u)+y\frac{\partial}{\partial y}(\operatorname{cosec} u)=\frac{1}{6}\operatorname{cosec} u$

$$\Rightarrow -x\operatorname{cosec} u\cot u\frac{\partial u}{\partial x}-y\operatorname{cosec} u\cdot\cot u\frac{\partial u}{\partial y}=\frac{1}{6}\operatorname{cosec} u$$

$$\Rightarrow \operatorname{cosec} u\cot u\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}=-\frac{1}{6}\operatorname{cosec} u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{6}\frac{1}{\cot u}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{6}\tan u \qquad \text{---(1)}$$

(ii) Differentiating (1) partially with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{6}\sec^2 u\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x dy}=-\left(1+\frac{1}{6}\sec^2 u\right)\frac{\partial u}{\partial x}$$

Multiply x on both sides, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x dy}=-x\left(1+\frac{1}{6}\sec^2 u\right)\frac{\partial u}{\partial x} \qquad \text{---(2)}$$

Differentiating (1) partial with respect to y, we get

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=-\frac{1}{6}\sec^2 u\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=-\left(1+\frac{1}{6}\sec^2 u\right)\frac{\partial u}{\partial y}$$

Multiply y on both sides, we get

$$xy\frac{\partial^2 u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-y\left(1+\frac{1}{6}\sec^2 u\right)\frac{\partial u}{\partial y} \qquad \text{---(3)}$$

Adding equations (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x dy}=-\left(1+\frac{1}{6}\sec^2 u\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=\frac{1}{6}\left(1+\frac{1}{6}\sec^2 u\right)\tan u \qquad \blacksquare$$

10. *If* $u=e^{\frac{x^2y^2}{x+y}}$, *prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u\log u$

➡ $u = e^{\frac{x^2 y^2}{x+y}}$

$$\log u = \frac{x^2 y^2}{x+y} = \frac{x^4\left(y^2/x^2\right)}{x(1+y/x)} = x^3\left[\frac{(y/x)^2}{1+y/x}\right] = x^3 g\left(\frac{y}{x}\right) = f \text{ (say)}$$

Therefore, *f* is homogenous function of degree 3. Hence by Euler's theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$$

$$x\frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) = 3\log u$$

$$x.\frac{1}{u}\frac{\partial u}{\partial x} + y.\frac{1}{u}\frac{\partial u}{\partial y} = 3\log u$$

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 3\log u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u$$

11. *If* $u = x^2 \tan^{-1}(y/x) - y^2 \tan(x/y)$, *show that*

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 2u$$

➡ Let $v = x^2 \tan^{-1}(y/x)$ ----(*a*), $w = y^2 \tan(x/y)$ ---(b)

i.e., $u = v - w$ ---(1)

By Euler's extension theorem (*a*) becomes,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 2(2-1)v = 2v \qquad \text{---(2)}$$

By Euler's extension theorem (*b*) becomes,

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 2(2-1)w = 2w \qquad \text{---(3)}$$

Now, (2) – (3) gives

$$x^2 \frac{\partial^2}{\partial x^2}(v-w) + 2xy\frac{\partial^2}{\partial x \partial y}(v-w) + y^2\frac{\partial^2}{\partial y^2}(v-w) = 2(v-w)$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u \qquad [\because u = v-w]$$ ■

Exercises

1. *Verify Euler's theorem in the following cases.*
   (i) $f = ax^2 + 2hxy + by^2$
   (ii) $f = \dfrac{xy}{x-y}$
   (iii) $f = x^n \log\left(\dfrac{y}{x}\right)$
   (iv) $f = x^3 + y^3 + 3x^2 y$
   (v) $f = \dfrac{1}{x^2 + y^2 + xy}$

2. *If* $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *prove that*
   (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$
   (ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{4}\left(\tan^3 u - \tan u\right)$

3. *If* $u = \tan^{-1}\dfrac{x^3+y^3}{x+y}$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

4. *If* $u = \sin^{-1}\left\{\left(x^2+y^2\right)(x+y)\right\}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$

5. *If* $u = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$