

$$\frac{t+2}{-4t+1} dt = dx$$

$$\int \frac{t+2}{1-4t} dt = \int dx + c \quad \text{---(2)}$$

$$-4t+1)t+2(-1/4$$

$$\frac{t-1/4}{9/4}$$

$$\therefore \frac{t+2}{1-4t} = \frac{1}{4} + \frac{9/4}{1-4t} = -\frac{1}{4} + \frac{9}{4(1-4t)} \quad \text{---(3)}$$

(2) becomes, $\int \left(-\frac{1}{4} + \frac{9}{4(1-4t)} \right) dt = \int dx + c$

$$-\frac{1}{4}t + \frac{9}{4} \frac{\log(1-4t)}{-4} = x + c$$

$$-\frac{t}{4} - \frac{9}{16} \log(1-4t) = x + c$$

$$-\frac{(2x-3y)}{4} - \frac{9}{16} \log(1-4(2x-3y)) - x = c$$

$$\frac{2x-3y}{4} + \frac{9}{16} \log(1-8x+12y) + x = -c$$

$$4(2x-3y) + 9 \log(1-8x+12y) + 16x = -16c$$

$$24x - 12y + 9 \log(1-8x+12y) = c_1 \quad \blacksquare$$

3. Solve $(2x+2y+5)dy = (x-y+3)dx$

▀ $(2x-2y+5)dy = (x-y+3)dx$

$$\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5} = \frac{x-y+3}{2(x-y)+5} \quad \text{---(1)}$$

Put $x-y=t \Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$

$$1 - \frac{dt}{dx} = \frac{dy}{dx}$$

(1) becomes, $1 - \frac{dt}{dx} = \frac{t+3}{2t+5}$

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$$\frac{dt}{dx} = 1 - \frac{t+3}{2t+5} = \frac{2t+5-(t+3)}{2t+5}$$

$$\frac{dt}{dx} = \frac{t+2}{2t+5}$$

$$\frac{2t+5}{t+2} dt = dx$$

$$\int \frac{2t+5}{t+2} dt = \int dx + c$$

$$\int \frac{2t+4+1}{t+2} dt = x + c$$

$$\int \frac{2(t+2)+1}{t+2} dt = x + c$$

$$\int \left(2 + \frac{1}{t+2} \right) dt = x + c$$

$$2t + \log(t+2) = x + c$$

$$2(x-y) + \log(x-y+2) - x = c$$

$$x - 2y + \log(x-y+2) = c \quad \blacksquare$$

4. Solve $(x+2y)(dx-dy) = dx+dy$

$$\Rightarrow (x+2y)(dx-dy) = dx+dy$$

$$(x+2y)dx - (x+2y)dy = dx+dy$$

$$(x+2y)dx - dx = dy + (x+2y)dy$$

$$(x+2y-1)dx = (x+2y+1)dy$$

$$\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1} \quad \text{---(1)}$$

$$\text{Put } x+2y = t \Rightarrow 1+2 \frac{dy}{dx} = \frac{dt}{dx}$$

$$2 \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{dt}{dx} - 1 \right)$$

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$$\begin{aligned} \therefore (1) \text{ becomes, } \quad \frac{1}{2} \left(\frac{dt}{dx} - 1 \right) &= \frac{t-1}{t+1} \\ \frac{dt}{dx} - 1 &= \frac{2(t-1)}{t+1} \\ \frac{dt}{dx} &= \frac{2t-2}{t+1} + 1 = \frac{2t-2+t+1}{t+1} \\ \frac{dt}{dx} &= \frac{3t-1}{t+1} \end{aligned}$$

$$\frac{t+1}{3t-1} dt = dx$$

$$\int \frac{t+1}{3t-1} dt = \int dx + c \quad \text{---(2)}$$

$$(3t-1) \left(t + \frac{1}{3} \right)$$

$$\frac{t-1/3}{4/3}$$

$$\frac{t+1}{3t-1} = \frac{1}{3} + \frac{4/3}{3t-1} = \frac{1}{3} + \frac{4}{3(3t-1)} \quad \text{---(3)}$$

Substituting (3) in (2), we get

$$\int \left(\frac{1}{3} + \frac{4}{3(3t-1)} \right) dt = \int dx + c$$

$$\frac{1}{3}t + \frac{4 \log(3t-1)}{3} = x + c$$

$$\frac{t}{3} + \frac{4}{9} \log(3t-1) = x + c$$

$$3t + 4 \log(3t-1) = 9x + 9c$$

$$3(x+2y) + 4 \log[3(x+2y)-1] = 9x + c_1, \text{ where } c_1 = 9c$$

$$6y - 6x + 4 \log[3x+6y-1] = c_1 \quad \blacksquare$$

5. Solve $\frac{dy}{dx} = \frac{3x+2y-5}{2x-3y+5}$

➔ $\frac{dy}{dx} = \frac{3x+2y-5}{2x-3y+5}$

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Put $x = X + h$ and $y = Y + k$

$$\therefore \frac{dy}{dx} = \frac{dY}{dX}$$

\therefore The given equation becomes,

$$\frac{dY}{dX} = \frac{3(X+h) + 2(Y+k) - 5}{2(X+h) - 3(Y+k) + 5}$$

$$\frac{dY}{dX} = \frac{3X + 2Y + 3h + 2k - 5}{2X - 3Y + 2h - 3k + 5} \quad \text{---(1)}$$

Now choose $3h + 2k - 5 = 0$, $2h - 3k + 5 = 0$

Solving the above equations, we get

$$h = \frac{5}{13}, \quad k = \frac{25}{13}$$

$$\therefore (1) \text{ becomes, } \frac{dY}{dX} = \frac{3X + 2Y}{2X - 3Y}$$

$$\text{Put } Y = VX \Rightarrow \frac{dY}{dX} = V + X \frac{dV}{dX}$$

$$\therefore V + X \frac{dV}{dX} = \frac{3X + 2VX}{2X - 3VX}$$

$$V + X \frac{dV}{dX} = \frac{X(3 + 2V)}{X(2 - 3V)}$$

$$V + X \frac{dV}{dX} = \frac{3 + 2V}{2 - 3V}$$

$$X \frac{dV}{dX} = \frac{3 + 2V}{2 - 3V} - V$$

$$= \frac{3 + 2V - 2V + 3V^2}{2 - 3V}$$

$$X \frac{dV}{dX} = \frac{3 + 3V^2}{2 - 3V} = \frac{3(1 + V^2)}{2 - 3V}$$

$$\frac{2 - 3V}{1 + V^2} dV = 3 \frac{dX}{X}$$

$$\int \left(\frac{2}{1 + V^2} - \frac{3V}{1 + V^2} \right) dV = 3 \int \frac{dX}{X} + c$$

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$$2 \tan^{-1} V - \frac{3}{2} \log(1+V^2) = 3 \log X + c$$

$$2 \tan^{-1} \left(\frac{Y}{X} \right) - \frac{3}{2} \log \left(1 + \frac{Y^2}{X^2} \right) = 3 \log X + c$$

$$2 \tan^{-1} \left(\frac{y-25/13}{x-5/13} \right) - \frac{3}{2} \log \left[1 + \frac{(y-25/13)^2}{(x-5/13)^2} \right] = 3 \log(x-5/13) + c \quad \blacksquare$$

6. Solve $(y-x+5)dx = (y+x+3)dy$

$$\Rightarrow (y-x+5)dx = (y+x+3)dy$$

$$\frac{dy}{dx} = \frac{y-x+5}{y+x+3}$$

$$\text{Put } y = Y + k, \quad x = X + h \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$$

$$\therefore \frac{dY}{dX} = \frac{(Y+k) - (X+h) + 5}{(Y+k) + (X+h) + 3}$$

$$\frac{dY}{dX} = \frac{Y-X+(k-h+5)}{Y+X+(k+h+3)}$$

$$\text{choose } k-h+5=0$$

$$k+h+3=0$$

Solving the above equations, we get

$$k = -4, \quad h = 1$$

$$\text{Therefore } \frac{dY}{dX} = \frac{Y-X}{Y+X}$$

$$\text{Put } Y = VX \Rightarrow \frac{dY}{dX} = V + X \frac{dV}{dX}$$

$$V + X \frac{dV}{dX} = \frac{VX - X}{VX + X}$$

$$V + X \frac{dV}{dX} = \frac{V-1}{V+1}$$

$$X \frac{dV}{dX} = \frac{V-1}{V+1} - V = \frac{V-1-V(V+1)}{V+1}$$

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$$X \frac{dV}{dX} = \frac{V-1-V^2-V}{V+1} = -\frac{1+V^2}{1+V}$$

$$\frac{1+V}{1+V^2} dV = -\frac{dX}{X}$$

$$\int \frac{1+V}{1+V^2} dV = -\int \frac{dX}{X} + c$$

$$\int \left(\frac{1}{1+V^2} + \frac{V}{1+V^2} \right) dV = -\log X + c$$

$$\tan^{-1} V + \frac{1}{2} \log(1+V^2) + \log X = c$$

$$2 \tan^{-1} V + \log(1+V^2) + 2 \log X = c_1, \text{ where } c_1 = 2c$$

$$2 \tan^{-1} \left(\frac{Y}{X} \right) + \log[(1+V^2)X^2] = c_1$$

$$2 \tan^{-1} \left(\frac{Y}{X} \right) + \log[X^2 + V^2 X^2] = c_1$$

$$2 \tan^{-1} \left(\frac{Y}{X} \right) + \log[X^2 + Y^2] = c_1$$

$$2 \tan^{-1} \left(\frac{y+4}{x-1} \right) + \log[(x-1)^2 + (y+4)^2] = c_1 \quad \blacksquare$$

Exercises

1) $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$

2) $(3x-2y+1)dy + (4y-6x-3)dx = 0$

3) $(2x+y+1)dx + (4x+2y-1)dy = 0$

4) $(2x+3y+4)dx + (4x+6y+5)dy = 0$

5) $(2x-4y+3)dy + (x-2y+1)dx = 0$

6) $\frac{dy}{dx} = \frac{6x-2y-7}{3x-y+4}$

7) $\frac{dy}{dx} = \frac{x+y-1}{x+y+1}$

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- 8) $(x - y - 2)dx - (2x - 2y - 3)dy = 0$
 9) $(2x + 3y - 1)dx = (2x + 3y + 2)dy$ given that $y = 3$ when $x = 1$
 10) $(4x - 6y - 1)dx - (2x - 3y + 2)dy = 0$
 11) $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$ 12) $\frac{dy}{dx} = \frac{x - y + 1}{x + 2y - 3}$
 13) $(2x + 9y - 20)dx - (6x + 2y - 10)dy = 0$
 14) $\frac{dy}{dx} = \frac{x - 1}{3x - 2y - 5}$
 15) $(3x - y - 1)dy + (2x - 2y)dx = 0$
 16) $(t - 8x + 7)dx = (t - x)dy$
 17) $(x - 2y + 5)dx - (2x + y - 1)dy = 0$
 18) $(5 + 3x - 4y)y' = 2 + 7x - 3y$
 19) $\frac{dy}{dx} = \frac{2x + 5y + 1}{5x + 2y - 1}$

Answers

- 1) $(y - x)^2 + 10y - 2x = c$ 2) $2x - y - \log(3x - 2y + 3) = c$
 3) $x + 2y - \log(6x + 3y - 3) = c$
 4) $21x + 42y - 9\log(14x + 21y + 22) = c$
 5) $4(x + 2y) - \log(4(x - 4y) + 5) = c$
 6) $2x - y - 15\log(3x - y + 15) = c$
 7) $\log(x + y) = x - y + c$
 8) $x - 2y - \log(2x - 2y - 1) = c$
 9) $x + y + 3\log(2x + 3y - 7) = 4 + 3\log 4$
 10) $x = -\frac{1}{4} \left[2x - 3y + \frac{15}{4} \log \left(2x + 3y - \frac{7}{4} \right) \right] + c$
 11) $3(2y - x) + \log(3x + 3y - 4) = c$
 12) $\left(x - \frac{1}{3} \right)^2 - 2 \left(x - \frac{1}{3} \right) \left(y - \frac{4}{3} \right) - 2 \left(y - \frac{4}{3} \right)^2 = c$

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- 13) $x + 2y - 5 = c(2x - y)^2$
 14) $(y - x + 2) = k(2y - x + 3)^2$
 15) $(x + y - 1)^4 = c(4x - 2y - 1)^2$
 16) $(8x^2 - 2tx - 14x + t^2 + 7) = c$
 17) $x^2 - y^2 - 4xy + 10x + 2y = c$
 18) $7x^2 + 4x - 6xy - 10y + 4y^2 = c$
 19) $(x + y)^7 = c\left(x - y - \frac{2}{3}\right)^3$

LINEAR DIFFERENTIAL EQUATIONS

The differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x), \quad \text{---(1)}$$

where $P(x)$ and $Q(x)$ are functions of x only is called linear differential equation in y

$$\text{OR} \quad \frac{dx}{dy} + P(y)x = Q(y) \quad \text{---(2)}$$

where $P(y)$ and $Q(y)$ are functions of y only is called linear differential equation in x .

If the differential equation can be put in the form $\frac{dy}{dx} + P(x)y = Q(x)$,

then multiply the given equation by $e^{\int P(x)dx}$, we get

$$e^{\int P dx} \frac{dy}{dx} + P y e^{\int P dx} = Q e^{\int P dx}$$

$$e^{\int P dx} \frac{dy}{dx} + y \frac{d}{dx} \left(e^{\int P dx} \right) = Q e^{\int P dx}$$

$$\text{i.e., } \frac{d}{dx} \left(y e^{\int P dx} \right) = Q e^{\int P dx}$$

Integrating, we get

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

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i.e., $y(I.F.) = \int Q(I.F.)dx + c$, where $I.F. = e^{\int Pdx}$
 which is the required general solution.

Working Rule

Case (i) If the differential equation is linear in y i.e., equation (1), then

Step (1) Find the integration factor (I.F.)

$$I.F. = e^{\int P(x)dx}$$

Step (2) Substitute IF in the equation

$$y(I.F.) = \int Q(I.F.)dx + c$$

On integrating the RHS of the above equation, we get the solution of the given differential equation.

Case (ii) If the differential equation is linear in x i.e., equation (2)

Step (1) Find the integration factor (I.F.)

$$I.F. = e^{\int P(y)dy}$$

Step (2) Substitute IF in the equation

$$x(I.F.) = \int Q(y)(I.F.)dy + c$$

On integrating the RHS of the above equation, we get the solution of the given differential equation.

Worked Examples

1. Solve $\frac{dy}{dx} + y \cot x = \sin x$

► $\frac{dy}{dx} + y \cot x = \sin x$, this is linear in y

$$P = \cot x \quad Q = \sin x$$

$$I.F. = e^{\int Pdx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

Therefore, the solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\begin{aligned} y(\sin x) &= \int (\sin x)(\sin x)dx + c \\ &= \int \sin^2 x dx + c \end{aligned}$$

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$$= \int \frac{1 - \cos 2x}{2} dx + c \quad \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx + c$$

$$= \frac{1}{2}x - \frac{1}{2} \frac{\sin 2x}{2} + c$$

$$y \sin x = \frac{1}{2}x - \frac{1}{4} \sin 2x + c \quad \blacksquare$$

2. Solve $x \log x \frac{dy}{dx} + y = 2 \log x$

▀ $x \log x \frac{dy}{dx} + y = 2 \log x$

Divide on both sides by $x \log x$, we get

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}, \text{ this is linear in } y$$

$$P = \frac{1}{x \log x} \quad Q = \frac{2}{x}$$

$$\text{I.F} = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx}$$

Put $\log x = t \Rightarrow (1/x) dx = dt$

$$\therefore \text{I.F} = e^{\int (1/t) dt} = e^{\log t} = t = \log x$$

Therefore, the solution of the given equation is

$$y(\text{I.F}) = \int Q(\text{I.F}) dx + c$$

$$y \log x = \int \frac{2}{x} \log x dx + c$$

Put $\log x = t \Rightarrow (1/x) dx = dt$

$$\therefore y \log x = 2 \int t dt + c = t^2 + c$$

$$\therefore y(\log x) = (\log x)^2 + c \quad \blacksquare$$

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