Proof Let $f(x) = A_o + A_1 x + A_2 x^2 + A_3 x^3 + ----$

Then by successive differentiation, we have

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + -----$$
$$f''(x) = 2A_2 + 3.2A_3 x + 4.3A_4 x^2 + -----$$
$$f'''(x) = 3.2.1A_3 + 4.3.2A_4 x + -------$$

Putting $x = 0$ in each of these, we get

$$f(0) = A_0,\ f'(0) = A_1$$
$$f''(0) = 2!A_2 \Rightarrow A_2 = \frac{f''(0)}{2!}$$
$$f'''(0) = 3!A_3 \Rightarrow A_3 = \frac{f'''(0)}{3!}$$

Therefore,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ---- + \frac{x^n}{n!}f^{(n)}(0) + ---$$

Worked Examples

1. *Expand* $\sin x$ *in powers of* $(x - \pi/2)$

➡ $f(x) = \sin x \Rightarrow f(\pi/2) = 1$

$$f'(x) = \cos x \Rightarrow f'(\pi/2) = 0$$
$$f''(x) = -\sin x \Rightarrow f''(\pi/2) = -1$$
$$f'''(x) = -\cos x \Rightarrow f'''(\pi/2) = 0$$
$$f^{iv}(x) = \sin x \Rightarrow f^{iv}(\pi/2) = 1$$

Therefore,

$$f(x) = f(\pi/2) + \frac{x - \pi/2}{1!}f'(\pi/2) + \frac{(x - \pi/2)^2}{2!}f''(\pi/2) + \frac{(x - \pi/2)^3}{3!}f'''(\pi/2) + ---$$

$$\sin x = 1 - \frac{(x - \pi/2)^2}{2!} + \frac{(x - \pi/2)^4}{4!} ---$$ ■

2. *Expand* $\log\sin x$ *in powers of* $x-2$

➡ $$f(x)=\log\sin x \Rightarrow f(2)=\log\sin 2$$

$$f'(x)=\frac{1}{\sin x}\cos x=\cot x \Rightarrow f'(2)=\cot 2$$

$$f''(x)=-\operatorname{cosec}^2 x \Rightarrow f''(2)=-\operatorname{cosec}^2 2$$

$$f'''(x)=2\operatorname{cosec}^2 x\cot x \Rightarrow f'''(2)=2\operatorname{cosec}^2 2\cot 2$$

Therefore,

$$f(x)=f(2)+\frac{x-2}{1!}f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+---$$

$$\log\sin x=\log\sin 2+(x-2)\cot 2-\frac{(x-2)^2}{2}\operatorname{cosec}^2 2$$

$$+\frac{(x-2)^3}{3}\operatorname{cosec}^2 2\cot 2+---$$ ■

3. *Expand* $\log x$ *in ascending powers of* $x-1$

➡ Let $y=\log x \Rightarrow y(1)=0$

$$y_1=\frac{1}{x} \Rightarrow y_1(1)=1$$

$$y_2=-1/x^2 \Rightarrow y_2(1)=-1$$

$$y_3=-1\left(-\frac{2}{x^3}\right)=\frac{2}{x^3} \Rightarrow y_3(1)=2$$

$$y_4=2\left(-\frac{3}{x^4}\right)=-\frac{6}{x^4} \Rightarrow y_4(1)=-6$$

Therefore,

$$y=y(1)+\frac{x-1}{1!}y_1(1)+\frac{(x-1)^2}{2!}y_2(1)+---$$

$$=0+\frac{(x-1)}{1!}(1)+\frac{(x-1)^2}{2!}(-1)+\frac{(x-1)^3}{3!}(2)+\frac{(x-1)^4}{4!}(-6)+---$$

$$\log x=\frac{x-1}{1}-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+---$$ ■

4. *Expand* $\log\cos x$ *in ascending powers of* $x-\pi/3$

➡ Let $y=\log\cos x \qquad \Rightarrow y\left(\frac{\pi}{3}\right)=\log\left(\frac{1}{2}\right)$

$$y_1=\frac{1}{\cos x}(-\sin x)=-\tan x \qquad \Rightarrow y_1\left(\frac{\pi}{3}\right)=-\sqrt{3}$$

$$y_2=-\sec^2 x \qquad \Rightarrow y_2\left(\frac{\pi}{3}\right)=-4$$

$$y_3=-2\cdot\sec x\cdot\sec x\tan x \qquad \Rightarrow y_3\left(\frac{\pi}{3}\right)=-8\sqrt{3}$$

Therefore,

$$y=y\left(\frac{\pi}{3}\right)+\frac{(x-\pi/3)}{1!}y_1\left(\frac{\pi}{3}\right)+\frac{(x-\pi/3)^2}{2!}y_2\left(\frac{\pi}{3}\right)+---$$

$$=\log\left(\frac{1}{2}\right)+\frac{(x-\pi/3)}{1!}\left(-\sqrt{3}\right)+\frac{(x-\pi/3)^2}{2!}(-4)+\frac{(x-\pi/3)^3}{3!}\left(-8\sqrt{3}\right)+---$$

$$\log\cos x=\log\left(\frac{1}{2}\right)-\sqrt{3}\left(x-\frac{\pi}{3}\right)-2\left(x-\frac{\pi}{3}\right)^2-\frac{4}{\sqrt{3}}\left(x-\frac{\pi}{3}\right)^3+---\blacksquare$$

5. *Expand* $\log(1+x)$ *by using Maclaurin's series*

➡ Let $f(x)=\log(1+x)\Rightarrow f(0)=0$

$$f'(x)=\frac{1}{1+x}\Rightarrow f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2}\Rightarrow f''(0)=-1$$

$$f'''(x)=\frac{2}{(1+x)^3}\Rightarrow f'''(0)=2$$

$$f^{iv}(x)=-\frac{6}{(1+x)^4}\Rightarrow f^{iv}(0)=-6$$

Therefore,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ---$$

$$= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + ---$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + --- \qquad \blacksquare$$

6. *Obtain the Maclaurin's series expansion of the function* e^x

➡ Let $y = e^x \Rightarrow y(0) = 1$

$y_1 = e^x \Rightarrow y_1(0) = 1$

$y_2 = e^x \Rightarrow y_2(0) = 1$

$y_3 = e^x \Rightarrow y_3(0) = 1$

$y_4 = e^x \Rightarrow y_4(0) = 1$

Therefore,

$$y = y(0) + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + ---$$

$$= 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + ---$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + --- \qquad \blacksquare$$

7. *Obtain the Maclaurin's series expansion of the function* $\sin x$

➡ Let $y = \sin x \Rightarrow y(0) = 0$

$y_1 = \cos x \Rightarrow y_1(0) = 1$

$y_2 = -\sin x \Rightarrow y_2(0) = 0$

$y_3 = -\cos x \Rightarrow y_3(0) = -1$

$y_4 = \sin x \Rightarrow y_4(0) = 0$

$y_5 = \cos x \Rightarrow y_5(0) = 1$

Therefore,

$$y = y(0) + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + ---$$

$$= 0 + \frac{x}{1!}(1) + 0 + \frac{x^3}{3!}(-1) + 0 + \frac{x^5}{5!}(1) + ---$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ---$$ ■

8. *Obtain the Maclaurin's series expansion of the function* $\sqrt{1+\sin 2x}$

➡

$$y = \sqrt{1+\sin 2x} = \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} = \sqrt{(\sin x + \cos x)^2}$$

$y = \sin x + \cos x \quad \Rightarrow y(0) = 1$

$y_1 = \cos x - \sin x \quad \Rightarrow y_1(0) = 1$

$y_2 = -\sin x - \cos x \quad \Rightarrow y_2(0) = -1$

$y_3 = -\cos x + \sin x \quad \Rightarrow y_3(0) = -1$

$y_4 = \sin x + \cos x \quad \Rightarrow y_4(0) = 1$

Therefore,

$$y = y(0) + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + ---$$

$$= 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(1) + ---$$

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + ---$$ ■

9. *Obtain the Maclaurin's series expansion of the function* $e^{\sin^{-1} x}$

➡ Let $y = e^{\sin^{-1} x} \quad \Rightarrow y(0) = 1$

$$y_1 = e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} \Rightarrow y_1(0) = 1$$

$$\sqrt{1-x^2}\, y_1 = e^{\sin^{-1} x}$$

$$\sqrt{1-x^2}\, y_1 = y$$

$$\left(1-x^2\right)y_1^2 = y^2$$
$$\left(1-x^2\right)2y_1y_2 + (-2x)y_1^2 = 2yy_1$$
$$\left(1-x^2\right)y_2 - xy_1 = y \qquad \text{---(1)}$$

At $x=0,\quad y_2(0)=1$

Differentiating (1) with respect to *x*, we get

$$\left(1-x^2\right)y_3 + (-2x)y_2 - xy_2 - y_1 = y_1$$
$$\left(1-x^2\right)y_3 - 3xy_2 - 2y_1 = 0 \qquad \text{---(2)}$$

At $x=0,\quad y_3(0)-2=0$

$$y_3(0)=2$$

Differentiating (2) with respect to *x*, we get

$$\left(1-x^2\right)y_4 + (-2x)y_3 - 3xy_3 - 3y_2 - 2y_2 = 0$$
$$\left(1-x^2\right)y_4 - 5xy_3 - 5y_2 = 0$$

At $x=0\quad y_4(0)-0-5(1)=0$

$$y_4(0)=5$$

Therefore,

$$y = y(0) + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + ---$$
$$= 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(5) + ---$$
$$e^{\sin^{-1}x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{5x^4}{24} + ---$$ ■

10. *Obtain the Maclaurin's series expansion of the function* $\log\sec x$

➨ Let $y = \log\sec x \Rightarrow y(0)=0$

$$y_1 = \frac{1}{\sec x}\sec x\tan x$$
$$y_1 = \tan x \Rightarrow y_1(0)=0$$
$$y_2 = \sec^2 x \Rightarrow y_2(0)=1$$
$$y_3 = 2\sec x.\sec x\tan x$$
$$y_3 = 2\sec^2 x\tan x \Rightarrow y_3(0)=0$$

$$y_3 = 2y_2 y_1 \qquad \text{since } y_2 = \sec^2 x, \quad y_1 = \tan x$$

$$y_4 = 2y_2 y_2 + 2y_3 y_1$$

$$y_4 = 2y_2^2 + 2y_1 y_3 \Rightarrow y_4(0) = 2$$

$$y_5 = 4y_2 y_3 + 2y_1 y_4 + 2y_2 y_3$$

$$y_5 = 6y_2 y_3 + 2y_1 y_4 \Rightarrow y_5(0) = 0$$

$$y_6 = 6y_2 y_4 + 6y_3 y_3 + 2y_2 y_4 + 2y_1 y_5$$

$$y_6 = 8y_2 y_4 + 6y_3^2 + 2y_1 y_5 \Rightarrow y_6(0) = 16$$

Therefore,

$$y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + ---$$

$$= 0 + 0 + \frac{x^2}{2!}(1) + 0 + \frac{x^4}{4!}(2) + 0 + \frac{x^6}{6!}(16) + ---$$

$$\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + ---$$ ■

11. *Obtain the Maclaurin's series expansion of the function* $\log(1+e^x)$

➡ Let $y = \log(1+e^x) \Rightarrow y(0) = \log 2$

$$y_1 = \frac{1}{1+e^x} e^x \Rightarrow y_1(0) = \frac{1}{2}$$

$$(1+e^x) y_1 = e^x$$

$$(1+e^x) y_2 + e^x y_1 = e^x \qquad ---(1)$$

At $x = 0$ $\quad 2y_2(0) + 1.y_1(0) = 1$

$$2y_2(0) + \frac{1}{2} = 1$$

$$y_2(0) = \frac{1}{4}$$

Differentiating (1) with respect to x, we get

$$(1+e^x) y_3 + e^x y_2 + e^x y_2 + e^x y_1 = e^x$$

$$\left(1+e^x\right)y_3+2e^x y_2+e^x y_1=e^x \qquad \text{---(2)}$$

At $x=0$, $\quad 2y_3(0)+2(1)\left(\frac{1}{4}\right)+1.\frac{1}{2}=1$

$$y_3(0)=0$$

Differentiating (2) with respect to x, we get

$$\left(1+e^x\right)y_4+e^x y_3+2e^x y_3+2e^x y_2+e^x y_2+e^x y_1=e^x$$

$$\left(1+e^x\right)y_4+3e^x y_3+3e^x y_2+e^x y_1=e^x$$

At $x=0$, $\quad 2y_4(0)+0+3(1)\left(\frac{1}{4}\right)+1\cdot\frac{1}{2}=1$

$$2y_4(0)+\frac{3}{4}+\frac{1}{2}=1$$

$$y_4(0)=-\frac{1}{8}$$

Therefore,

$$y=y(0)+\frac{x}{1!}y_1(0)+\frac{x^2}{2!}y_2(0)+---$$

$$=\log 2+\frac{x}{1!}\left(\frac{1}{2}\right)+\frac{x^2}{2!}\left(\frac{1}{4}\right)+0+\frac{x^4}{4!}\left(-\frac{1}{8}\right)+---$$

$$=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+--- \qquad ■$$

12. *Expand* $\log(1+\cos x)$ *by Maclaurin's series upto the term containing* x^4

➡ Let $y=\log(1+\cos x) \Rightarrow y(0)=\log 2$

$$y_1=\frac{1}{1+\cos x}(-\sin x)=-\frac{\sin x}{1+\cos x} \Rightarrow y_1(0)=0$$

$$(1+\cos x)y_1+\sin x=0$$

Differentiating with respect to x, we get

$$(1+\cos x)y_2+(-\sin x)y_1+\cos x=0$$

$$(1+\cos x)y_2-y_1\sin x+\cos x=0 \qquad \text{---(1)}$$

At $x=0,\quad 2y_2(0)-0+1=0 \Rightarrow y_2(0)=-\dfrac{1}{2}$

Differentiating (1) with respect to x, we get

$$(1+\cos x)y_3+(-\sin x)y_2-y_2\sin x-y_1\cos x-\sin x=0$$

$$(1+\cos x)y_3-2y_2\sin x-y_1\cos x-\sin x=0 \qquad \text{---(2)}$$

At $x=0,\quad 2y_3(0)-0-0-0=0 \Rightarrow y_3(0)=0$

Differentiating (2) with respect to x, we get

$$(1+\cos x)y_4+(-\sin x)y_3-2y_3\sin x-2y_2\cos x-y_2\cos x + y_1\sin x-\cos x=0$$

$$(1+\cos x)y_4-3y_3\sin x-3y_2\cos x+y_1\sin x-\cos x=0$$

At $x=0,\quad 2y_4(0)-0-3\left(-\dfrac{1}{2}\right)(1)+0-1=0$

$$2y_4(0)+\frac{3}{2}-1=0$$

$$2y_4(0)+\frac{1}{2}=0$$

$$y_4(0)=-\frac{1}{4}$$

Therefore,

$$y=y(0)+\frac{x}{1!}y_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+---$$

$$=\log 2+\frac{x}{1!}(0)+\frac{x^2}{2!}\left(-\frac{1}{2}\right)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}\left(-\frac{1}{4}\right)+---$$

$$\log(1+\cos x)=\log 2-\frac{x^2}{4}-\frac{x^4}{96}+---$$ ■

13. *Obtain the Maclaurin's series expansion of the function* $e^x\sin x$

➡ Let $y=e^x\sin x \Rightarrow y(0)=0$

$$y_1=e^x\cos x+e^x\sin x \Rightarrow y_1(0)=1$$

$$y_1=e^x(\cos x+\sin x)$$

$$y_2=e^x(-\sin x+\cos x)+e^x(\cos x+\sin x)$$

$$= e^x(-\sin x + \cos x + \cos x + \sin x)$$

$$y_2 = 2e^x \cos x \Rightarrow y_2(0) = 2$$

$$y_3 = 2e^x(-\sin x) + 2e^x \cos x = 2e^x(\cos x - \sin x) \Rightarrow y_3(0) = 2$$

$$y_4 = 2e^x(-\sin x - \cos x) + 2e^x(\cos x - \sin x)$$

$$y_4 = -4e^x \sin x \Rightarrow y_4(0) = 0$$

$$y_5 = -4e^x \cos x - 4e^x \sin x \Rightarrow y_5(0) = -4$$

Therefore,

$$y = y(0) + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + ----$$

$$= 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(-4) + ----$$

$$e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + ----$$ ■

14. *Expand* $\tan^{-1}(1+x)$ *as for as the term containing* x^3, *using Maclaurin's series*

➡ Let $y = \tan^{-1}(1+x) \Rightarrow y(0) = \pi/4$

$$y_1 = \frac{1}{1+(1+x)^2} = \frac{1}{x^2+2x+2} \Rightarrow y_1(0) = \frac{1}{2}$$

$$y_2 = -\frac{1}{(x^2+2x+2)^2}(2x+2)$$

$$y_2 = \frac{-2(x+1)}{(x^2+2x+2)^2} \Rightarrow y_2(0) = -\frac{1}{2}$$

$$y_3 = -2\left\{\frac{(x^2+2x+2)^2(1) - (x+1)2(x^2+2x+2)(2x+2)}{(x^2+2x+2)^4}\right\}$$

$$y_3 = -2\left\{\frac{x^2+2x+2-2(x+1)(2x+2)}{(x^2+2x+2)^3}\right\}$$

$$y_3(0) = -\left\{\frac{2-4}{2^3}\right\} = \frac{1}{4}$$

Therefore,

$$y = y(0) + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + ---$$

$$= \frac{\pi}{4} + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(-\frac{1}{2}\right) + \frac{x^3}{3!}\left(\frac{1}{4}\right) + ---$$

$$y = \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{24} + ---$$

Exercises

I.

1) *Expand* $\sin x$ *in ascending powers of* $(x - \pi/2)$
2) *Expand* $\log x$ *in powers of* $(x-2)$
3) Obtain Taylor's expansion of $\tan^{-1} x$ in powers of $(x-1)$ up to the term containing $(x-1)^3$
4) *Expand* $\log\sin x$ *in ascending powers of* $(x-3)$
5) *Expand* $\log x$ *in powers of* $(x-1)$
6) *Obtain Taylor's series for the function* $\cos x$ *about* $\pi/4$

II. *Obtain the Maclaurin's expansions of the following functions*

1) $\sec x$ 2) $\sin^{-1} x$ 3) $\log(1+\cos x)$
4) $\log(1+\tan x)$ 5) $e^{\cos x}$
6) $e^{\tan^{-1} x}$ 7) $e^{x\cos x}$ 8) $\sec^2 x$
9) $e^x \sec x$ 10) $\log(1+\sin x)$
11) $\log(1+e^x)$ 12) $\log(\sec x + \tan x)$
13) $\log\cos x$ 14) $e^x \cos x$ 15) $e^{\sin x}$

Answers

I.

1) $\sin x = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 + ---$

2) $\log x = \log 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2) + \frac{1}{24}(x-3)^3 + ---$

3) $\tan^{-1} x = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$

4) $\log \sin x = \log \sin 3 - (x-3)\cot 3 - \frac{(x-3)^2}{2!}\operatorname{cosec}^2 3$

$$+ 2\frac{(x-3)}{3!}\operatorname{cosec}^2 3 \cot 3 + ---$$

5) $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + ---$

6) $\cos x = \frac{1}{\sqrt{2}}\left\{1 - \frac{(x-\pi/4)}{1\,!} - \frac{(x-\pi/4)^2}{2\,!} + \frac{(x-\pi/4)^3}{3\,!} + ---\right\}$

II.

1) $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + ---$

2) $x + \frac{x^3}{6} + \frac{3x^5}{40} + ---$

3) $\log 2 - \frac{x^2}{4} - \frac{x^4}{96}$

4) $x - \frac{x^2}{2!} + \frac{4x^3}{3!}$

5) $e\left\{1 - \frac{x^2}{2} + \frac{x^4}{6} + ---\right\}$

6) $1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} + ---$

7) $1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} + ---$

8) $1 + x^2 + \frac{2}{3}x^4 + ---$

9) $1 + x + x^2 + \frac{2}{3}x^3 + ---$

10) $1 - \frac{x^2}{2} + \frac{x^3}{6} + ---$

11) $\log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{196} + ---$

12) $x+\frac{x^3}{6}+\frac{x^5}{24}+---$ 13) $-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}+---$

14) $1+x-\frac{2x^3}{3!}-\frac{4x^4}{4!}+\frac{4x^5}{5!}+---$

15) $1+x+\frac{x^2}{2}-\frac{x^4}{8}+---$

PARTIAL DIFFERENTIATION

Functions of Two or More Variables

So far, we have been dealing with the functions of a single independent variable. We often come across quantities, which depend on two or more variables. For example, the volume (*V*) of a gas is a function of pressure (*P*) and temperature (*T*); i.e., $V = V(P,T)$

Partial Derivatives

Consider a function $f(x,y)$ which depends on two independent variables *x* and *y*, denoted by $f = f(x,y)$. The partial derivative of $f(x,y)$ with respect to *x* is the ordinary derivative of $f(x,y)$ when y is regarded as a constant. It is written as $\partial f/\partial x$ or $D_x f$ or f_x

Thus, $\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$

Similarly, the partial derivative of $f(x,y)$ with respect to *y* is the ordinary derivative of $f(x,y)$ when *x* is regarded as a constant.

Thus, $\frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$

The partial derivatives of f_x and f_y are $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

or $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y\partial x}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y^2}$ respectively

It should be noted that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \text{ and } \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

The student will be able to convince himself/herself that in all ordinary cases,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Worked Examples

1. *If* $z = x^y + y^x$ *prove that* $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

➡ $z = x^y + y^x$

Differentiating partially with respect to x, we get

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

Next, differentiating partially with respect to y, we get

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = y\left(x^{y-1}\log x\right) + 1 \cdot x^{y-1} + y^x \cdot \frac{1}{y} + \log y\left(xy^{x-1}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = x^{y-1}\left[y\log x + 1\right] + y^{x-1}\left[1 + x\log y\right] \qquad \text{---(1)}$$

Now differentiating z partially with respect to y, we get

$$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$

Differentiating this partially with respect to x, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^y \frac{1}{x} + \log x\left(yx^{y-1}\right) + xy^{x-1}\log y + y^{x-1} \cdot 1$$

$$\frac{\partial^2 z}{\partial x \partial y} = x^{y-1}\left[1 + y\log x\right] + y^{x-1}\left[x\log y + 1\right] \qquad \text{---(2)}$$

From equation (1) and (2), we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$ ■

2. *If* $z = e^{ax+by} f(ax - by)$, *prove that* $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

➡ Let $ax + by = u$ $\qquad v = ax - by$

$$\frac{\partial u}{\partial x} = a, \quad \frac{\partial u}{\partial y} = b \qquad \frac{\partial v}{\partial x} = a, \quad \frac{\partial v}{\partial y} = -b$$

Therefore, $z = e^u f(v)$

$$\frac{\partial z}{\partial x} = e^u \frac{\partial u}{\partial x} f(v) + e^u . f'(v)\frac{\partial v}{\partial x} = e^u (a) f(v) + e^u f'(v)(a)$$

$$\frac{\partial z}{\partial x} = ae^u [f(v) + f'(v)] \qquad \text{---(1)}$$

$$\frac{\partial z}{\partial y} = e^u . f'(v)\frac{\partial v}{\partial y} + e^u \frac{\partial u}{\partial y}.f(v) = e^u f'(v)(-b) + e^u f(v) b.$$

$$\frac{\partial z}{\partial y} = e^u b[f(v) - f'(v)] \qquad \text{---(2)}$$

$$\therefore b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^u [f(v) + f'(v)] + abe^u [f(v) - f'(v)]$$

$$= abe^u [f(v) + f'(v) + f(v) - f'(v)] = abe^u [2f(v)]$$

$$= 2abz \qquad [\because z = e^u f(v)]$$ ■

3. *If* $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, *prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

➡ $$u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y+z} + y\left(\frac{-1}{(z+x)^2}\right) + z\left(\frac{-1}{(x+y)^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y+z} - \frac{y}{(z+x)^2} - \frac{z}{(x+y)^2}$$

Similarly, $$\frac{\partial u}{\partial y} = -\frac{x}{(y+z)^2} + \frac{1}{z+x} - \frac{z}{(x+y)^2}$$

$$\frac{\partial u}{\partial z} = -\frac{x}{(y+z)^2} - \frac{y}{(z+x)^2} + \frac{1}{(x+y)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y+z} - \frac{xy}{(z+x)^2} - \frac{xz}{(x+y)^2} - \frac{xy}{(y+z)^2} + \frac{y}{z+x}$$

$$- \frac{yz}{(x+y)^2} - \frac{xz}{(y+z)^2} - \frac{yz}{(z+x)^2} + \frac{z}{(x+y)}$$

$$= \left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) - \left(\frac{y(z+x)}{(z+x)^2} + \frac{z(x+y)}{(x+y)^2} + \frac{x(y+z)}{(y+z)^2}\right)$$

$$= \left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) - \left(\frac{y}{(z+x)} + \frac{z}{(x+y)} + \frac{x}{(y+z)}\right) = 0$$ ■

4. *If* $u(x+y) = x^2 + y^2$*, then prove that*

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

➡ $u(x+y) = x^2 + y^2$

$$\Rightarrow u = \frac{x^2 + y^2}{x+y} \quad \text{---(1)}$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = \frac{(x+y)(2y)-(x^2+y^2)}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \left[\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2 = \left[\frac{2(x+y)(x-y)}{(x+y)^2}\right]^2$$

$$= \frac{4(x-y)^2}{(x+y)^2} \qquad \text{---(2)}$$

$$4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right) = 4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2-(x^2+2xy-y^2)-(y^2+2xy-x^2)}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2-4xy}{(x+y)^2}\right] = \frac{4(x-y)^2}{(x+y)^2} \qquad \text{---(3)}$$

Therefore, $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)$ ■

5. *If* $z = 2(ax+by)^2 - (x^2+y^2)$ *and* $a^2+b^2=1$, *then prove that* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

➡ $z = 2(ax+by)^2 - (x^2+y^2)$ ---(1)

Differentiating partially with respect to x, we get

$$\frac{\partial z}{\partial x} = 4(a)(ax+by)-(2x) = 4a^2x + 4aby - 2x$$

Differentiating again, we get

$$\frac{\partial^2 z}{\partial x^2} = 4a^2 - 2$$

Now differentiating (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = 4(ax+by)(b)-(2y) = 4abx + 4b^2y - 2y$$

Differentiating again, we get

$$\frac{\partial^2 z}{\partial y^2} = 4b^2 - 2$$

Therefore,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4a^2 - 2 + 4b^2 - 2$$

$$= 4(a^2+b^2) - 4 = 4(1) - 4 = 0 \quad ■$$

6. *If* $u = x^2y + y^2z + z^2x$, *prove that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2$

➡ $u = x^2y + y^2z + z^2x$ ---(1)

Differentiating (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = 2xy + z^2 \qquad \text{---(2)}$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = 2yz + x^2 \qquad \text{---(3)}$$

Differentiating (1) partially with respect to z, we get

$$\frac{\partial u}{\partial z} = 2zx + y^2 \qquad \text{---(4)}$$

Adding equations (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy + z^2 + 2yz + x^2 + 2zx + y^2$$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}=(x+y+z)^2$$ ■

7. *If* $u=\frac{1}{r}$ *where* $r^2=(x-a)^2+(y-b)^2+(z-c)^2$ *prove that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

➡ $u=\frac{1}{r}$ ---(1)

Where $r^2=(x-a)^2+(y-b)^2+(z-c)^2$

$$2r\cdot\frac{\partial r}{\partial x}=2(x-a)\Rightarrow\frac{\partial r}{\partial x}=\frac{(x-a)}{r}$$

Similarly, $\frac{\partial r}{\partial y}=\frac{(y-b)}{r}$ $\qquad \frac{\partial r}{\partial z}=\frac{(z-c)}{r}$

Differentiating (1) with respect to x partially, we get

$$\frac{\partial u}{\partial x}=-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}=-\frac{1}{r^2}\cdot\frac{x-a}{r}=-\frac{1}{r^3}(x-a)$$

Differentiating again with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{r^6}\left[r^3(1)-(x-a)(3r^2)\cdot\left(\frac{\partial r}{\partial x}\right)\right]=-\frac{1}{r^6}\left[r^3-(x-a)3r^2\frac{(x-a)}{r}\right]$$

$$=-\frac{1}{r^6}\left[r^3-3r(x-a)^2\right]=-\frac{1}{r^5}\left[r^2-3(x-a)^2\right] \quad ---(2)$$

Similarly, $\frac{\partial^2 u}{\partial y^2}=-\frac{1}{r^5}\left[r^2-3(y-b)^2\right]$ ---(3)

$$\frac{\partial^2 u}{\partial z^2}=-\frac{1}{r^5}\left[r^2-3(z-c)^2\right] \quad ---(4)$$

Adding equations (2), (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{r^2-3(x-a)^2}{r^5}-\frac{r^2-3(y-b)^2}{r^5}-\frac{r^2-3(z-c)^2}{r^5}$$

$$=-\frac{1}{r^5}\left[r^2-3(x-a)^2+r^2-3(y-b)^2+r^2-3(z-c)^2\right]$$

$$= -\frac{1}{r^5}\left\{3r^2 - 3\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]\right\}$$

$$= -\frac{1}{r^5}\left(3r^2 - 3r^2\right) = 0 \qquad \blacksquare$$

8. *If* $u = (y-z)(z-x)(x-y)$ *show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

➡ $u = (y-z)(z-x)(x-y)$ ---(1)

Differentiating (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = (y-z)\{(z-x)1 + (-1)(x-y)\}$$

$$= (y-z)(z-x) - (y-z)(x-y) \qquad \text{---(2)}$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = (z-x)(y-z)(-1) + (z-x)(x-y).1$$

$$= -(z-x)(y-z) + (z-x)(x-y) \qquad \text{---(3)}$$

Differentiating (1) partially with respect to z, we get

$$\frac{\partial u}{\partial z} = (x-y)(y-z)1 + (x-y)(z-x)(-1)$$

$$= (x-y)(y-z) - (x-y)(z-x) \qquad \text{---(4)}$$

Adding equations (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (y-z)(z-x) - (y-z)(x-y)$$

$$- (z-x)(y-z) + (z-x)(x-y)$$

$$+ (x-y)(y-z) - (x-y)(z-x)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \qquad \blacksquare$$

9. *If* $u = (x-y)^n + (y-z)^n + (z-x)^n$, *show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

➡ $u = (x-y)^n + (y-z)^n + (z-x)^n$ ---(1)

Differentiating (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = n(x-y)^{n-1} + n(z-x)^{n-1}(-1) = n(x-y)^{n-1} - n(z-x)^{n-1} \quad \text{---(2)}$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = n(x-y)^{n-1}(-1) + n(y-z)^{n-1} = -n(x-y)^{n-1} + n(y-z)^{n-1} \quad \text{---(3)}$$

Differentiating (1) partially with respect to z, we get

$$\frac{\partial u}{\partial z} = n(y-z)^{n-1}(-1) + n(z-x)^{n-1} = -n(y-z)^{n-1} + n(z-x)^{n-1} \quad \text{---(4)}$$

Adding equations (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = n\Big[(x-y)^{n-1} - (z-x)^{n-1} - (x-y)^{n-1} + (y-z)^{n-1} - (y-z)^{n-1} + (z-x)^{n-1}\Big]$$

$$= n[0] = 0$$

■

10. *If* $u = f(r)$ *and* $r^2 = x^2 + y^2$ *show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

$$r^2 = x^2 + y^2$$

$$2r.\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Given $u = f(r)$

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{x}{r} f'(r) = \frac{x f'(r)}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[r\frac{\partial}{\partial x}(x f'(r)) - x f'(r)\frac{\partial r}{\partial x}\right]$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left\{r\left[x f''(r)\frac{\partial r}{\partial x} + f'(r)\right] - x f'(r)\frac{\partial r}{\partial x}\right\}$$

$$= \frac{1}{r^2}\left[r x f''(r).\frac{x}{r} + r f'(r) - x f'(r)\frac{x}{r}\right]$$

$$= \frac{1}{r^2}\left\{x^2 f''(r) + f'(r)\left[r - \frac{x^2}{r}\right]\right\} \qquad \text{---(1)}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left\{y^2 f''(r) + f'(r)\left[r - \frac{y^2}{r}\right]\right\} \qquad \text{---(2)}$

Adding equations (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{1}{r^2}\left\{x^2 f''(r) + f'(r)\left[r - \frac{x^2}{r}\right]\right\} + \frac{1}{r^2}\left\{y^2 f''(r) + f'(r)\left[r - \frac{y^2}{r}\right]\right\}$$

$$= \frac{1}{r^2}\left\{f''(r)\left(x^2 + y^2\right) + f'(r)\left[r - \frac{x^2}{r} + r - \frac{y^2}{r}\right]\right\}$$

$$= \frac{1}{r^2}\left\{f''(r)\left(r^2\right) + f'(r)\left[2r - \frac{r^2}{r}\right]\right\}$$

$$= f''(r) + \frac{1}{r} f'(r)$$

11. *If $u = r^m$, where $r^2 = x^2 + y^2 + z^2$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$$

➡ $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{2x}{2r} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$u = r^m,$

Differentiating with respect to x, we get

$$\therefore \frac{\partial u}{\partial x} = mr^{m-1} \cdot \frac{\partial r}{\partial x} = mr^{m-1} \cdot \frac{x}{r} = mr^{m-2} \cdot x,$$