$$= \frac{1}{16}\left[z^4 + 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} + 4z \cdot \frac{1}{z^3} + \frac{1}{z^4} \right]$$

$$= \frac{1}{16}\left[z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2} \right) + 6 \right]$$

$$= \frac{1}{16}\left[2\cos 4\theta + 4 \cdot 2\cos 2\theta + 6 \right]$$

$$= \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3 \right)$$ ■

22. Prove that $\sin^7\theta = -\frac{1}{64}\left(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta \right)$

➡ Let $z = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$

$$z^n = \cos n\theta + i\sin n\theta, \quad \frac{1}{z^n} = \cos n\theta - i\sin n\theta$$

$$\Rightarrow z - \frac{1}{z} = 2i\sin\theta \text{ and } z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Therefore, $\sin^7\theta = \left[\frac{1}{2i}\left(z - \frac{1}{z} \right) \right]^7$

$$= \frac{1}{2^7 i^7}\left(z - \frac{1}{z} \right)^7$$

$$= \frac{1}{-128i}\left[z^7 - 7z^5 + 21z^3 - 35z + \frac{35}{z} - \frac{21}{z^3} + \frac{7}{z^5} - \frac{1}{z^7} \right]$$

$$= \frac{i}{128}\left[\left(z^7 - \frac{1}{z^7} \right) - 7\left(z^5 - \frac{1}{z^5} \right) + 21\left(z^3 - \frac{1}{z^3} \right) - 35\left(z - \frac{1}{z} \right) \right]$$

$$= \frac{i}{128}\left[2i\sin 7\theta - 7 \cdot 2i\sin 5\theta + 21 \cdot 2i\sin 3\theta - 35 \cdot 2i\sin\theta \right]$$

$$= \frac{2i^2}{128}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta \right]$$

$$\sin^7\theta = -\frac{1}{64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta \right]$$ ■

23. Express $\sin^4\theta\cos^3\theta$ in terms of cosines of multiples of θ

➡ Let $z=\cos\theta+i\sin\theta$

$$\frac{1}{z}=\cos\theta-i\sin\theta$$

$$z^n=\cos n\theta+i\sin n\theta$$

$$\frac{1}{z^n}=\cos n\theta-i\sin n\theta$$

$$z^n+\frac{1}{z^n}=2\cos n\theta$$

$$z^n-\frac{1}{z^n}=2i\sin n\theta$$

$$z+\frac{1}{z}=2\cos\theta$$

$$z-\frac{1}{z}=2i\sin\theta$$

$$\sin^4\theta\cos^3\theta=\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^4\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^3$$

$$=\frac{1}{128}\left(z-\frac{1}{z}\right)\left(z-\frac{1}{z}\right)^3\left(z+\frac{1}{z}\right)^3$$

$$=\frac{1}{128}\left(z-\frac{1}{z}\right)\left[z^2-\frac{1}{z^2}\right]^3$$

$$=\frac{1}{128}\left(z-\frac{1}{z}\right)\left[\left(z^2\right)^3-3\left(z^2\right)^2\frac{1}{z^2}+3z^2\left(\frac{1}{z^2}\right)^2-\left(\frac{1}{z^2}\right)^3\right]$$

$$=\frac{1}{128}\left(z-\frac{1}{z}\right)\left[z^6-3\cdot z^2+\frac{3}{z^2}-\frac{1}{z^6}\right]$$

$$=\frac{1}{128}\left[z^7-3\cdot z^3+\frac{3}{z}-\frac{1}{z^5}-z^5+3\cdot z-\frac{3}{z^3}+\frac{1}{z^7}\right]$$

$$=\frac{1}{128}\left[\left(z^7+\frac{1}{z^7}\right)-3\left(z^3+\frac{1}{z^3}\right)-\left(z^5+\frac{1}{z^5}\right)+3\left(z+\frac{1}{z}\right)\right]$$

$$= \frac{1}{128}\left[2\cos 7\theta - 3(2\cos 3\theta) - 2\cos 5\theta + 3(2\cos\theta)\right]$$

$$= \frac{1}{64}\left[\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta\right] \quad \blacksquare$$

Exercises

I. *Simplify the following using De Moivre's theorem*

(1) $(\cos 2\theta + i\sin 2\theta)^4 (\cos 3\theta - i\sin 3\theta)^2$

(2) $\left(\cos\frac{5\theta}{2} + i\sin\frac{5\theta}{2}\right)^4$

(3) $\dfrac{(\cos 3\theta + \sin 3\theta)(\cos 2\theta - i\sin 2\theta)}{(\cos 4\theta + i\sin 4\theta)^{-4}\cos(5\theta + i\sin 5\theta)^2}$

(4) $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{30}$

(5) $\dfrac{(\cos 4\theta + i\sin 4\theta)^3}{(\sin 3\theta + i\cos 3\theta)^2 (\cos 5\theta + i\sin 5\theta)^3}$

(6) $(\sin\theta + i\cos\theta)^n$

(7) $\dfrac{(\cos 3\alpha + \sin 3\alpha)^2 (\cos 2\beta - i\sin 2\beta)}{(\cos 3\alpha - i\sin 3\gamma)^4}$

(8) $(\cos\theta + i\sin\theta)^5 (\cos\theta - i\sin\theta)$

(9) $\dfrac{(\cos\theta + i\sin\theta)^7}{(\cos\theta - i\sin\theta)^4}$

(10) $\dfrac{(\sin\theta + i\cos\theta)^3}{(\cos\alpha + i\sin\alpha)}$

II. *Prove the following*

(1) $\left(i + \sqrt{3}\right)^4 - \left(-i + \sqrt{3}\right)^4 = 16\sqrt{3}i$

(2) $\left(1 + i\sqrt{3}\right)^5 + \left(1 - i\sqrt{3}\right)^5 = 32$

(3) $(-1 + i)^7 + (-1 - i)^7 = -16$

III. *If n is a positive integer, prove the following*

(1) $\left(1+i\sqrt{3}\right)^n + \left(1-i\sqrt{3}\right)^n = 2^{n+1}\cos\frac{(n\pi)}{3}$

(2) $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\left(\frac{n\pi}{4}\right)$

(3) $\left(\sqrt{3}+i\right)^n + \left(\sqrt{3}-i\right)^n = 2^{n+1}\cos\left(\frac{n\pi}{6}\right)$

(4) $(1+\cos\theta+i\sin\theta)^n + (1+\cos\theta-i\sin\theta)^n$

$$= 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$$

(5) $\left(-1+i\sqrt{3}\right)^{3n} + \left(-1-i\sqrt{3}\right)^{3n} = 2^{3n+1}$

IV. *If* $\cos\alpha+\cos\beta+\cos\gamma = 0$ *and* $\sin\alpha+\sin\beta+\sin\gamma = 0$, *prove that,*

(1) $(\cos 3\alpha+\cos 3\beta+\cos 3\gamma) = 3\cos(\alpha+\beta+\gamma)$

(2) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(3) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \sin^2\alpha+\sin^2\beta+\sin^2\gamma = 3/2$

(4) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma = 0$

V. *If* $\cos\alpha+2\cos\beta+3\cos\gamma = 0$ *and* $\sin\alpha+2\sin\beta+3\sin\gamma = 0$ *prove that*

(1) $\sin 3\alpha+8\sin 3\beta+27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$

(2) $\cos 3\alpha+8\cos 3\beta+27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

VI. (1) *Prove that*

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

(2) *Prove that* $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n = \cos n\theta + i\sin n\theta$

(3) $\left(\frac{1-\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\right) = \cos n(\pi-\theta)+i\sin n(\pi-\theta)$

(4) $\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \frac{1+i\tan n\theta}{1-i\tan n\theta}$, where n is a positive integer.

(5) *Prove that* $\left(\frac{1-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}}\right)^{10} = -1$

(6) *Prove that* $\left(\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right)^{\frac{1}{2}} = -1$

VII. (1) *If* $x = \cos\theta + i\sin\theta$*, then show that*

$x^2 + \frac{1}{x^2} = 2\cos 2\theta$ *and* $x^3 - \frac{1}{x^3} = 2i\sin 3\theta$

(2) *If* $x = \cos\theta + i\sin\theta$ *and* $y = \cos\varphi + i\sin\varphi$,

then prove that $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta-\varphi)$ *and*

$\frac{x}{y} - \frac{y}{x} = 2i\sin(\theta-\varphi)$

(3) *If* $x = \cos\alpha + i\sin\alpha$ *and* $y = \cos\beta + i\sin\beta$,
then prove the following

(i) $xy + \frac{1}{xy} = 2\cos(\alpha+\beta)$ (ii) $xy - \frac{1}{xy} = 2i\sin(\alpha+\beta)$

(iii) $x^2 y^3 + \frac{1}{x^2 y^3} = 2\cos(2\alpha + 3\beta)$

VIII. *If* $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$
then prove the following.

(1) $xyz + \frac{1}{xyz} = 2\cos(\alpha+\beta+\gamma)$

(2) $\frac{xy}{z} + \frac{z}{xy} = 2\cos(\alpha+\beta-\gamma)$

(3) $\dfrac{x^l\ y^n}{z^n} + \dfrac{z^n}{x^l y^m} = 2\cos\left(l\alpha + m\beta - n\gamma\right)$

(4) $\sqrt{\dfrac{x}{yz}} + \sqrt{\dfrac{yz}{x}} = 2\cos(\alpha - \beta - \gamma)$

IX. (1) *If* $x + \dfrac{1}{x} = 2\cos\theta$*, prove that* $x = \cos\theta \pm i\sin\theta$

(2) *If* $x + \dfrac{1}{x} = 2\cos\alpha$ *and* $y + \dfrac{1}{y} = 2\cos\beta$*, then prove the following*

(i) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m} = 2\cos(m\alpha - n\beta)$

(ii) $\dfrac{x^n}{y^m} - \dfrac{y^m}{x^n} = 2i\sin(n\alpha - m\beta)$

(iii) $3\sqrt{\dfrac{x}{y}} + 3\sqrt{\dfrac{y}{x}} = 2\cos\ (\alpha - \beta)$

Answers

I. (1) $\cos 2\theta + i\sin 2\theta$ (2) $\cos 10\theta + i\sin 10\theta$

(3) $\cos(24\theta) + i\sin(24\theta)$ (4) 1

(5) $-\cos 15\theta - i\sin 15\theta$

(6) $\cos n\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)$

(7) $\cos(6\alpha - 6\beta + 12\gamma) + i\sin(6\alpha - 6\beta + 12\ \gamma)$

(8) $\cos\theta + i\sin\theta$ (9) $\cos 11\theta + i\sin 11\theta$

(10) $-\left[\sin(3\theta + 2\alpha) + i\cos(3\theta + 2\alpha)\right]$

n^{th} ROOTS OF A COMPLEX NUMBER

Let $z = x + iy$ be a complex number and $z = r(\cos\theta + i\sin\theta)$ be its polar form.

Where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore n^{th}$ root of a complex number is

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$$

Write the general value of the amplitude

$$\theta + 2k\pi, k \in I$$

$$\therefore (z)^{\frac{1}{n}} = (r)^{\frac{1}{n}}[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]^{\frac{1}{n}}$$

By using De Moivre's theorem, we get

$$z_k = z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$\Rightarrow z_k = r^{\frac{1}{n}}\ \text{cis}\left(\frac{\theta + 2k\pi}{n}\right) \qquad k = 0, 1, 2 ----(n-1)$$

This gives n values which are the n-distinct n^{th} roots of given complex number.

Worked Examples

1. *Find the cube roots of* $1 - i$

Let $z = x + iy = 1 - i$

$$\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{1}\right| = \frac{\pi}{4}.$$

Here $z = 1 - i = (1, -1)$ is in the 4^{th} quadrant

$$\theta = 2\pi - \alpha = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Therefore, $z=\sqrt{2}\left[\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)\right]$

or $\quad z=\sqrt{2}\left[\cos\left(2k\pi+\frac{7\pi}{4}\right)+i\sin\left(2k\pi+\frac{7\pi}{4}\right)\right]$

$$z=2^{\frac{1}{2}}\left[\cos\left(\frac{8k+7}{4}\right)\pi+i\sin\left(\frac{8k+7}{4}\right)\pi\right]$$

$$\Rightarrow z^{\frac{1}{3}}=2^{\frac{1}{6}}\left[\cos\left(\frac{8k+7}{4}\right)\pi+i\sin\left(\frac{8k+7}{4}\right)\pi\right]^{\frac{1}{3}}$$

$$z_k=z^{\frac{1}{3}}=2^{\frac{1}{6}}\left[\cos\left(\frac{8k+7}{12}\right)\pi+i\sin\left(\frac{8k+7}{12}\right)\pi\right] \text{ where } k=0,1,2$$

$$z_0=2^{\frac{1}{6}}\left[\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right]$$

$$z_1=2^{\frac{1}{6}}\left[\cos\frac{15\pi}{12}+i\sin\frac{15\pi}{12}\right]$$

$$z_2=2^{\frac{1}{6}}\left[\cos\frac{23\pi}{12}+i\sin\frac{23\pi}{12}\right]$$

2. *Find the fourth roots of $1-i\sqrt{3}$ and represent them on an Argand plane.*

➡ Let $z=1-i\sqrt{3}=x+iy$

$$r=\sqrt{x^2+y^2}=\sqrt{1+3}=2$$

$$\alpha=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{-\sqrt{3}}{1}\right|=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$$

Here $z=1-i\sqrt{3}=(1,-\sqrt{3})$ is in the 4th quadrant

$$\theta=2\pi-\alpha=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$$

Therefore, $z=2\left[\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right]$

or $$z = 2\left[\cos\left(2k\pi + \frac{5\pi}{3}\right) + i\sin\left(2k\pi + \frac{5\pi}{3}\right)\right]$$

$$z = 2\left[\cos\left(\frac{6k+5}{3}\right)\pi + i\sin\left(\frac{6k+5}{3}\right)\pi\right]$$

Therefore, $$z_k = z^{\frac{1}{4}} = 2^{\frac{1}{4}}\left[\cos\left(\frac{6k+5}{3}\right)\pi + i\sin\left(\frac{6k+5}{3}\right)\pi\right]^{\frac{1}{4}}$$

$$z_k = 2^{\frac{1}{4}}\left[\cos\left(\frac{6k+5}{12}\right)\pi + i\sin\left(\frac{6k+5}{12}\right)\pi\right] \text{ where } k = 0,1,2,3$$

$$z_0 = 2^{\frac{1}{4}}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$$

$$z_1 = 2^{\frac{1}{4}}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right]$$

$$z_2 = 2^{\frac{1}{4}}\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right]$$

$$z_3 = 2^{\frac{1}{4}}\left[\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right]$$

These are the 4 roots of given complex number. To represents those on argand plane, draw a circle with centered at origin and radius $2^{\frac{1}{4}}$ and mark all the 4 roots on the circle.

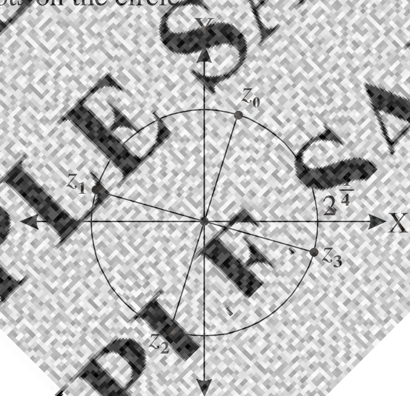

Figure 1.10

■

3. *Find the fourth roots of* $-1+i\sqrt{3}$ *and represent them on the argand diagram.*

➡ Let $z=-1+i\sqrt{3}=x+iy \Rightarrow r=\sqrt{1+3}=2$

$$\alpha=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{\sqrt{3}}{-1}\right|=\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}$$

Here $z=-1+i\sqrt{3}=\left(-1,\sqrt{3}\right)$ is in the 2^{nd} quadrant.

$$\theta=\pi-\alpha=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

Therefore, $z=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$$=2\left[\cos\left(2k\pi+\frac{2\pi}{3}\right)+i\sin\left(2k\pi+\frac{2\pi}{3}\right)\right]$$

$$z=2\left[\cos\left(\frac{6k+2}{3}\right)\pi+i\sin\left(\frac{6k+2}{3}\right)\pi\right]$$

$$z_k=z^{\frac{1}{4}}=2^{\frac{1}{4}}\left[\cos\left(\frac{6k+2}{3}\right)\pi+i\sin\left(\frac{6k+2}{3}\right)\pi\right]^{\frac{1}{4}}$$

$$=2^{\frac{1}{4}}\left[\cos\left(\frac{6k+2}{12}\right)\pi+i\sin\left(\frac{6k+2}{12}\right)\pi\right]$$

$$z_k=2^{\frac{1}{4}}\left[\cos\left(\frac{3k+1}{6}\right)\pi+i\sin\left(\frac{3k+1}{6}\right)\pi\right] \text{ where } k=0,1,2,3$$

$$z_0=2^{\frac{1}{4}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$z_1 = 2^{\frac{1}{4}}\left(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right) = 2^{\frac{1}{4}}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$z_2 = 2^{\frac{1}{4}}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$z_3 = 2^{\frac{1}{4}}\left(\cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}\right) = 2^{\frac{1}{4}}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

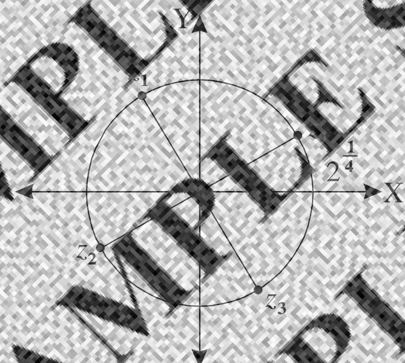

Figure 1.

4. *Solve* $z^8 + 1 = 0$

➡ $z^8 + 1 = 0 \Rightarrow z^8 = -1 = x + iy$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 0} = 1$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{0}{-1}\right| = 0$$

Here -1= $(-1,0)$ is in the 2^{nd} quadrant

$$\theta = \pi - \alpha = \pi$$

Therefore, $z^8 = 1(\cos\pi + i\sin\pi)$

$$z^8 = \cos(2k\pi + \pi) + \sin(2k\pi + \pi)$$

$$z_K = z = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{\frac{1}{8}}$$

$$z_k = \cos\left(\frac{2k+1}{8}\right)\pi + i\sin\left(\frac{2k+1}{8}\right)\pi \quad \text{where } k = 0, 1, 2, 3, 4, 5, 6, 7$$

$$z_0 = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)$$

$$z_1 = \cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)$$

$$z_2 = \cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)$$

$$z_3 = \cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)$$

$$z_4 = \cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)$$

$$z_5 = \cos\left(\frac{11\pi}{8}\right) + i\sin\left(\frac{11\pi}{8}\right)$$

$$z_6 = \cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)$$

$$z_7 = \cos\left(\frac{15\pi}{8}\right) + i\sin\left(\frac{15\pi}{8}\right)$$

$$\therefore z = z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_7$$ ■

5. *Find all the values of* $\left(1 + i\sqrt{3}\right)^{\frac{2}{3}}$

➡ Let $z = 1 + \sqrt{3}i = x + iy$

$$\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \frac{\pi}{3}$$

Here $z = 1 + i\sqrt{3} = (1, \sqrt{3})$ is in the 1st quadrant

$$\theta = \alpha = \frac{\pi}{3}$$

Therefore, $z = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

$$= 2\left[\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right]$$

$$z = 2\left[\cos\left(\frac{6k+1}{3}\right)\pi + i\sin\left(\frac{6k+1}{3}\right)\pi\right]$$

$$z^{\frac{2}{3}} = 2^{\frac{2}{3}}\left[\cos\left(\frac{6k+1}{3}\right)\pi + i\sin\left(\frac{6k+1}{3}\right)\pi\right]^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\frac{2}{3}\left(\frac{6k+1}{3}\right)\pi + i\sin\frac{2}{3}\left(\frac{6k+1}{3}\right)\pi\right]$$

$$z^{\frac{2}{3}} = 2^{\frac{2}{3}}\left[\cos\left(\frac{12k+2}{9}\right)\pi + i\sin\left(\frac{12k+2}{9}\right)\pi\right] \text{ where } k = 0, 1, 2$$

$$z_0 = 2^{\frac{2}{3}}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) = 2^{\frac{2}{3}}\text{ cis }\frac{2\pi}{9}$$

$$z_1 = 2^{\frac{2}{3}}\left[\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right] = 2^{\frac{2}{3}}\text{ cis }\frac{14\pi}{9}$$

$$z_2 = 2^{\frac{2}{3}}\left[\cos\frac{26\pi}{9} + i\sin\frac{26\pi}{9}\right] = 2^{\frac{2}{3}}\text{ cis }\frac{26\pi}{9}$$ ■

6. *The centre of a regular hexagon is at the origin and one vertex is given by* $\sqrt{3} + i$ *on the Argand diagram. Determine the other vertex.*

➡

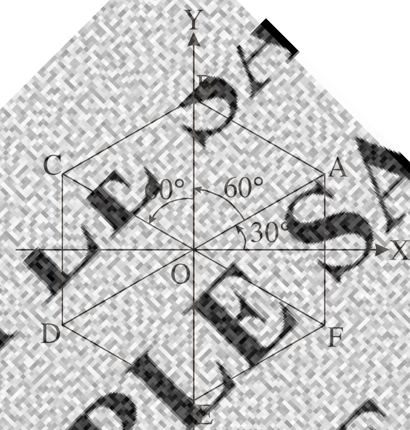

Figure 1.12

Let $A=\sqrt{3}+i$

$$OA=\sqrt{3+1}=2$$

$$X\hat{O}A=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=30°$$

Therefore,

$$X\hat{O}B=30°+60°=90°$$

$$X\hat{O}C=90°+60°=150°$$

$$X\hat{O}D=150°+60°=210°$$

$$X\hat{O}E=210°+60°=270°$$

$$X\hat{O}F=270°+60°=330°$$

Hence,

$$B=2(\cos 90°+i\sin 90°)=2i$$

$$C=2(\cos 150°+i\sin 150°)=2\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=-\sqrt{3}+i$$

$$D=2(\cos 210°+i\sin 210)=2\left(-\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right)=-\sqrt{3}-i$$

$$E = 2(\cos 270° + i\sin 270°) = 2(0 + i(-1)) = -2i$$

$$F = 2(\cos 330° + i\sin 330°) = 2\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = \sqrt{3} - i$$

Exercises

I. *Find the cube roots of the following complex numbers and represent them on the argand plane*

(1) $\sqrt{3} - i$ (2) $-1-i$ (3) $\frac{1-i\sqrt{3}}{2}$ (4) $\frac{-1+i}{2}$

(5) 8 (6) $-2i$

II. *Find the fourth roots of the following complex numbers and represent them on the argand plane*

(1) $\sqrt{3}+i$ (2) $1-i$

(3) $\frac{-1-i\sqrt{3}}{2}$ (4) $8+8i\sqrt{2}$

III. *Find all the values of the following and find their continued product.*

(1) $(\sqrt{3}+i)^{\frac{1}{3}}$ (2) $\left(\frac{-1-i\sqrt{3}}{2}\right)^{\frac{3}{4}}$

(3) $27^{\frac{1}{3}}$ (4) $(8i)^{\frac{1}{3}}$

(5) $(\sqrt{3}+i)^{\frac{4}{3}}$ (6) $(1+i)^{\frac{5}{4}}$

Answers

I (1) $2^{\frac{1}{3}}\,\text{cis}\left(\frac{12k+11}{18}\right)\pi \quad k = 0, 1, 2$

(2) $(2)^{\frac{1}{6}}\,\text{cis}\left(\frac{8k+5}{12}\right)\pi \quad k = 0, 1, 2$

(3) $\text{cis}\left(\frac{6k+5}{9}\right)\pi \quad k = 0, 1, 2$

(4) $\left(\frac{1}{2}\right)^{\frac{1}{6}} \text{cis}\left(\frac{8k+3}{12}\right)\pi \quad k = 0, 1, 2$

(5) $2\text{cis}\left(\frac{2k\pi}{3}\right) \quad k = 0, 1, 2$

(6) $2^{\frac{1}{3}} \text{cis}\left(\frac{4k+3}{6}\right)\pi \quad k = 0, 1, 2$

II (1) $2^{\frac{1}{4}}\text{cis}\left(\frac{12k+1}{24}\right)\pi \quad k = 0, 1, 2, 3$

(2) $2^{\frac{1}{8}} \text{cis}\left(\frac{8k+7}{16}\right)\pi \quad k = 0, 1, 2, 3$

(3) $\text{cis}\left(\frac{3k+2}{6}\right)\pi \quad k = 0, 1, 2, 3$

(4) $2\text{cis}\left(\frac{6k+1}{12}\right)\pi \quad k = 0, 1, 2, 3$

III (1) $2^{\frac{1}{3}}\text{cis}\left(\frac{12k+1}{18}\right)\pi \quad k = 0, 1, 2$ Product $= \sqrt{3} + i =$ P

(2) $\text{cis}\frac{K\pi}{2} \quad k = 0, 1, 2, 3 \quad P = -1$

(3) $3\text{cis}\left(\frac{2k\pi}{3}\right) \quad k = 0, 1, 2 \quad P = 27$

(4) $2\text{cis}\left(\frac{4k+1}{6}\right)\pi \quad k = 0, 1, 2, \quad P = 8i$

(5) $2^{\frac{4}{3}} \text{cis}\left(\frac{12k+11}{9}\right)\pi \quad k = 0, 1, 2, \quad P = 8\left(-1 - i\sqrt{3}\right)$

(6) $2^{\frac{5}{8}} \text{cis}\left(\frac{8k+1}{16}\right)\pi \quad k = 0, 1, 2, 3, \quad P = 4(1+i)$