$$x^2 + x - 2 = 0$$
$$x^2 + 2x - x - 2 = 0$$
$$x(x+2) - 1(x+2) = 0$$
$$(x-1)(x+2)$$
$$x = 1, -2$$

If $x = 1 \Rightarrow y = 1$, if $x = -2 \Rightarrow y = 4$

Here $x : -2 \to 1,\ y : x^2 \to 2 - x$

$$\therefore \iint_R dxdy = \int_{-2}^{1} \int_{x^2}^{2-x} dydx$$
$$= \int_{-2}^{1} y\Big]_{x^2}^{2-x} dx = \int_{-2}^{1} \left[(2-x) - x^2\right] dx$$
$$= 2x - \frac{x^2}{2} - \frac{x^3}{3}\Bigg|_{-2}^{1}$$
$$= 2(1-(-2)) - \frac{1}{2}\left(1-(-2)^2\right) - \frac{1}{3}\left(1-(-2)^3\right)$$
$$= 2(3) - \frac{1}{2}(-3) - \frac{1}{3}(9)$$
$$= 6 + 3/2 - 3 - = 3 + 3/2 = 9/2$$ ■

16. *Evaluate* $\iint xy\ dxdy$ *over the region in positive quadrant for which* $x + y \le 1$

➡

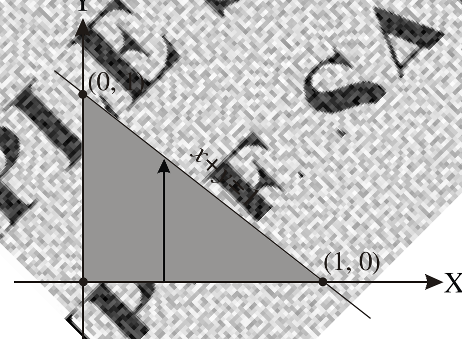

Figure 3.9

$x : 0 \to 1,\ y : 0 \to (1-x)$

$$I = \iint xy\ dxdy = \int_0^1 \int_0^{(1-x)} xy\ dy\ dx$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{(1-x)} dx = \int_0^1 \left[x\frac{(1-x)^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[x\left(1-2x+x^2\right)\right] dx = \frac{1}{2}\int_0^1 \left(x-2x^2+x^3\right) dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{24}$$ ■

Exercises

I. Evaluate the following integrals

1) $\int_0^1 \int_{x^2}^{x} dydx$

2) $\int_1^2 \int_y^{3y} (x+y)dxdy$

3) $\int_0^\infty \int_y^\infty xe^{-\frac{x^2}{y}} dxdy$

4) $\int_0^\pi \int_0^{\cos\theta} r\sin\theta\, drd\theta$

5) $\int_0^1 \int_0^{1-x} xydydx$

6) $\int_1^2 \int_0^x \frac{dydx}{x^2+y^2}$

7) $\int_0^1 \int_0^{x^2} xe^y dydx$

8) $\int_0^1 \int_0^{\sqrt{x}} \left(x^2+y^2\right)dydx$

9) $\int_0^\pi \int_0^{2a\cos\theta} r^3 drd\theta$

10) $\int_0^\pi \int_0^{\sin y} dxdy$

11) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dxdy$ 12) $\displaystyle\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\,dydx$

13) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dxdy$ 14) $\displaystyle\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dxdy$

15) $\displaystyle\int_0^{\pi/2} \int_0^{\infty} \frac{r\,dr\,d\theta}{\left(r^2+a^2\right)^2}$ 16) $\displaystyle\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1+x^2+y^2}$

II 1) *Evaluate* $\iint_A x^2 y^2\,dxdy$, *where A is the region bounded by* $x=0,\quad y=0$ *and* $x^2+y^2=1$

2) *Evaluate* $\int_A xy(x+y)\,dA$, *where A is the area bounded between the parabola* $y=x^2$ *and the line* $y=x$

3) *Evaluate* $\iint_A y\,dxdy$, *where A is the region enclosed between the parabolas* $y^2=4x$ *and* $x^2=4y$

4) *If A is the triangular area with vertices* (0,0), (10,1) *and* (1,1), *evaluate* $\iint_A \sqrt{xy-y^2}\,dxdy$

5) *If A is the area bounded by the lines* $y=0$ *and* $x=1$ *and the parabola* $y=x^2$, *then evaluate* $\iint_A xy^2\,dxdy$

6) *Evaluate* $\iint_A e^{2x+3y}\,dxdy$ *over the region A bounded by* $x=0$, $y=0$ *and* $x+y=1$

7) *Evaluate* $\iint_R \left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{1/2} dxdy$ *where R is the region bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

8) *Evaluate* $\iint_A y\,dxdy$, *where A is the region in the first quadrant bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

9) *If R is the region bounded by the lines* $x=2,\ y=1$ *and the parabola* $y=x^2$, *show that* $\iint_R \left(x^2+y^2\right)dxdy=\frac{1006}{105}$

10) *If R is the triangular bounded by the x – axis, y – axis and the line* $\frac{x}{a}+\frac{y}{b}=1$, *show that* $\iint_R xy\,dxdy=\frac{a^2b^2}{24}$

11) *If R is the region bounded by the circle* $x^2+y^2=1$ *in the first quadrant, evaluate* $\iint_R \frac{xy}{\sqrt{1-y^2}}dxdy$

Answers

I. 1) $\frac{1}{6}$ 2) 14 3) $\frac{1}{2}$

4) $\frac{1}{3}$ 5) $\frac{1}{24}$ 6) $\frac{\pi}{4}\log 2$

7) $\frac{1}{2}(e-2)$ 8) $\frac{3}{35}$ 9) $3\pi\frac{a^4}{4}$

10) 2 11) $\frac{a^5}{15}$ 12) $\frac{1}{2}$

13) $\frac{\pi a^3}{6}$ 14) $\frac{16}{3}a^2$ 15) $\frac{\pi}{4a^2}$

16) $\frac{\pi}{4}\log\left(1+\sqrt{2}\right)$

II. 1) $\frac{\pi}{96}$ 2) $\frac{3}{56}$ 3) $\frac{48}{5}$

4) 6   5) $\frac{1}{24}$ 6) $\frac{1}{6}(e-1)^2(2e+1)$

7) $\frac{2\pi ab}{3}$ 8) $\frac{ab^2}{3}$ 11) $\frac{1}{6}$

TRIPLE INTEGRALS

Triple integral can be evaluated by expressing it in terms of three integrals in the form

$$I = \iiint_v f(x, y, z)\,dxdydz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dzdydx$$

If x_1, x_2 are constants; y_1, y_2 are constants or functions of x; z_1, z_2 are constants or functions of x and y then the above integral is evaluated as follows:

First $f(x, y, z)$ is integrated with respect to z between z_1 and z_2 keeping x and y fixed. The resulting expression is then integrated with respect to y treating x as constant between y_1 and y_2. The result thus obtained is finally integrated with respect to x from x_1 to x_2

i.e., $\int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left\{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz\right\}dy\right]dx$

where the integration is carried out from the innermost bracket to the outermost bracket.

Note Evaluation of the integral may be performed in any order if all the limits are constants.

Worked Examples

1. *Evaluate* $\int_0^1\int_0^1\int_0^1 (x + y + z)\,dxdydz$.

➡ $\int_0^1\int_0^1\int_0^1 (x + y + z)\,dxdydz = \int_0^1\int_0^1 \frac{x^2}{2} + (y + z)x\Big|_0^1 dydz$