$$\frac{1+v}{1+v^2} = -\frac{dx}{x}$$

$$\int\left(\frac{1}{1+v^2}+\frac{v}{1+v^2}\right)dv = -\int\frac{dx}{x}+c$$

$$\tan^{-1}v+\frac{1}{2}\log\left(1+v^2\right) = -\log x + c$$

$$2\tan^{-1}v+\log\left(1+v^2\right) = -2\log x + 2c$$

$$2\tan^{-1}\left(\frac{y}{x}\right)+\log\left(1+\frac{y^2}{x^2}\right)+\log x^2 = 2c$$

$$2\tan^{-1}\left(\frac{y}{x}\right)+\log\left[x^2\left(1+\frac{y^2}{x^2}\right)\right] = c_1$$

$$2\tan^{-1}\left(\frac{y}{x}\right)+\log\left(x^2+y^2\right) = c_1$$

11. *Solve* $\frac{dy}{dx} = \frac{y}{x-\sqrt{xy}}$

$$\frac{dy}{dx} = \frac{y}{x-\sqrt{xy}} \qquad \text{---(1)}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

(1) becomes, $v + x\frac{dv}{dx} = \frac{vx}{x-\sqrt{x(vx)}} = \frac{vx}{x-x\sqrt{v}}$

$$v + x\frac{dv}{dx} = \frac{v}{1-\sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{v}{1-\sqrt{v}} - v = \frac{v-v\left(1-\sqrt{v}\right)}{1-\sqrt{v}}$$

$$= \frac{v-v+v\sqrt{v}}{1-\sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{v\sqrt{v}}{1-\sqrt{v}}$$

$$\frac{1-\sqrt{v}}{v\sqrt{v}}\,dv = \frac{dx}{x}$$

$$\left(\frac{1}{v\sqrt{v}} - \frac{\sqrt{v}}{v\sqrt{v}}\right)dv = \int \frac{dx}{x} + c$$

$$\int\left(v^{-\frac{3}{2}} - \frac{1}{v}\right)dv = \log x + c$$

$$\frac{v^{-\frac{1}{2}}}{-(1/2)} - \log v = \log x + c$$

$$-\frac{2}{\sqrt{v}} = \log v + \log x + c$$

$$-\frac{2}{\sqrt{y/x}} = \log(vx) + c$$

$$-2\sqrt{\frac{x}{y}} = \log y + c$$

$$\log y + 2\sqrt{\frac{x}{y}} + c = 0$$

12. *Solve* $[x + y\tan(x/y)]dy - ydx = 0$

➡ $[x + y\tan(x/y)]dy - ydx = 0$

$$[x + y\tan(x/y)]dy = ydx$$

$$\frac{dx}{dy} = \frac{x + y\tan(x/y)}{y}$$

$$\frac{dx}{dy} = \frac{x}{y} + \tan\left(\frac{x}{y}\right) \qquad \text{---(1)}$$

Put $\frac{x}{y} = v$ i.e., $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

(1) becomes, $\quad v + y\frac{dv}{dy} = v + \tan v$

$$y\frac{dv}{dy} = \tan v$$

$$\frac{dv}{\tan v} = \frac{dy}{y}$$

$$\Rightarrow \int \frac{dv}{\tan v} = \int \frac{dy}{y} + c$$

$$\int \cot v\,dv = \int \frac{1}{y}\,dy + c$$

$$\log(\sin v) = \log y + \log c_1$$

$$\log(\sin v) = \log(c_1 y)$$

$$\sin v = c_1 y$$

$$\sin(x/y) = c_1 y$$

13. *Solve* $x\frac{dy}{dx} = y + x\cos^2\left(\frac{y}{x}\right)$

➡ $x\frac{dy}{dx} = y + x\cos^2\left(\frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right) \quad \text{---(1)}$$

Put $\frac{y}{x} = v$ i.e. $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

(1) becomes, $\quad v + x\frac{dv}{dx} = v + \cos^2 v$

$$x\frac{dv}{dx} = \cos^2 v$$

$$\frac{dv}{\cos^2 v} = \frac{dx}{x}$$

$$\sec^2 v\,dv = \frac{dx}{x}$$

$$\int \sec^2 v\,dv = \int \frac{1}{x}\,dx + c$$

$$\tan v = \log x + c$$

$$\tan\left(\frac{y}{x}\right) = \log x + c \qquad \blacksquare$$

14. *Solve* $\left(x^3 + y^3\right)dx = \left(x^2 y + xy^2\right)dy$

➡ $\left(x^3 + y^3\right)dx = \left(x^2 y + xy^2\right)dy$

$$\frac{dy}{dx} = \frac{x^3 + y^3}{x^2 y + xy^2} \qquad \text{---(1)}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

(1) becomes, $v + x\frac{dv}{dx} = \frac{x^3 + (vx)^3}{x^2(vx) + x(vx)^2}$

$$v + x\frac{dv}{dx} = \frac{1 + v^3}{v + v^2}$$

$$x\frac{dv}{dx} = \frac{1 + v^3}{v + v^2} - v = \frac{1 + v^3 - v\left(v + v^2\right)}{v + v^2}$$

$$= \frac{1 - v^2}{v(1 + v)} = \frac{(1 - v)(1 + v)}{v(1 + v)}$$

$$x\frac{dv}{dx} = \frac{1 - v}{v}$$

$$\frac{v}{1 - v}dv = \frac{dx}{x}$$

$$\int\frac{v - 1 + 1}{1 - v}dv = \int\frac{dx}{x} + c$$

$$\int\left(-1 + \frac{1}{1 - v}\right)dv = \log x + c$$

$$-v + \frac{\log(1 - v)}{-1} = \log x + c$$

$$-\frac{y}{x} - \log\left(1 - \frac{y}{x}\right) = \log x + c \qquad \blacksquare$$

15. *Solve* $\left(x^2 + y^2\right)\dfrac{dy}{dx} = xy$

➡ $\left(x^2 + y^2\right)\dfrac{dy}{dx} = xy$

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} \qquad \text{---(1)}$$

Put $y = vx \Rightarrow v + x\dfrac{dv}{dx} = \dfrac{dy}{dx}$

$\therefore$ (1) becomes, $v + x\dfrac{dv}{dx} = \dfrac{x(vx)}{x^2 + (vx)^2} = \dfrac{v}{1+v^2}$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v = \frac{v - v - v^3}{1+v^2}$$

$$x\frac{dv}{dx} = -\frac{v^3}{1+v^2}$$

$$\frac{1+v^2}{v^3}\,dv = -\frac{dx}{x}$$

$$\int\left(v^{-3} + \frac{1}{v}\right)dv = -\int\frac{1}{x}\,dx + c$$

$$\frac{v^{-2}}{-2} + \log v = -\log x + c$$

$$-\frac{1}{2v^2} + \log v + \log x = c$$

$$-\frac{1}{2(y/x)^2} + \log(vx) = c$$

$$-\frac{x^2}{2y^2} + \log y = c$$

■

Exercises

1) $x\dfrac{dy}{dx} + x - y + x\operatorname{cosec}\left(\dfrac{y}{x}\right) = 0$ 2) $x\dfrac{dy}{dx} = y + xe^{\frac{y}{x}}$

3) $x\cos(y/x)(ydx + xdy) = y\sin(y/x)(xdy - ydx)$

4) $(x+y)\dfrac{dy}{dx} = x - y$

5) $xy\dfrac{dy}{dx} = x^2 - xy + y^2$

6) $(x+y)dx - (x-y)dy = 0$

7) $x^2\dfrac{dy}{dx} - 3xy - 2y^2 = 0$

8) $x\dfrac{dy}{dx} = y + 2xe^{-\frac{y}{x}}$

9) $y^2dx + (x^2 - xy + y^2)dy = 0$

10) $(x^2 - xy + y^2)dx - xydy = 0$

11) $xdy - ydx = \sqrt{x^2 + y^2}\,dx$

12) $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0$

13) $x(x-y)dy + y^2dx = 0$

14) $2xy\dfrac{dy}{dx} = x^2 + 3y^2$

15) $x^2dy + y(x+y)dx = 0$

16) $x^3dx - y^3dy = 3xy(ydx - xdy)$

Answers

1) $\sec\left(\dfrac{y}{x}\right) - \tan\left(\dfrac{y}{x}\right) + \dfrac{y}{x} + \log x = c$

2) $\log x + e^{-\frac{y}{x}} = c$

3) $xy\cos\left(\dfrac{y}{x}\right) = c$

4) $x^2 - 2xy - y^2 = c$

5) $(x-y) = ce^{-\frac{y}{x}}$

6) $\tan^{-1}\dfrac{y}{x} = \log\sqrt{x^2 + y^2} + c$

7) $y = cx^2(x+y)$

8) $e^{\frac{y}{x}} = \log cx^2$

9) $\tan^{-1}\left(\dfrac{y}{x}\right) = \log y + c$

10) $\log(y-x) = c - \left(\dfrac{y}{x}\right)$

11) $y + \sqrt{x^2 + y^2} = cx^2$

12) $x + ye^{\frac{x}{y}} = c$

13) $y = ce^{(y/x)}$

14) $x^2 + y^2 = cx^3$

15) $y + 2x = cx^2y$

16) $(x^2 + y^2)^4 = c(y^2 - x^2)^2$

EQUATIONS REDUCIBLE TO HOMOGENEOUS

Some first order ODEs are not homogeneous but can be put in homogeneous form by a simple transformation. Suppose the equation is of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

Where a, b, c, A, B, C are constants. This equation is not homogeneous but can be *reduced to the variable separable form* or *to the homogeneous form* by transformations.

Case (i) If $ax + by = m(Ax + By)$ where m is constant, then

Put $Ax + By = t \Rightarrow A + B\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dy}{dx} = \frac{1}{B}\left(\frac{dt}{dx} - A\right)$$

Substituting the above substitutions in the given equation, we get

$$\frac{dy}{dx} = \frac{m(Ax + By) + c}{Ax + By + C}$$

$$\frac{1}{B}\left(\frac{dt}{dx} - A\right) = \frac{mt + c}{t + C}$$

$$\frac{dt}{dx} = A + B\left(\frac{mt + c}{t + C}\right)$$

which is a variable separable form.

Case (ii) *If* $ax + by \neq m(Ax + By)$, then

Put $x = X + h$ and $y = Y + k$

$$\Rightarrow dx = dX \qquad dy = dY$$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Using the above transformation in the given differential equation, we get

$$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{A(X + h) + B(Y + k) + C}$$

$$\frac{dY}{dX} = \frac{(aX + bY) + ah + bk + c}{(AX + BY) + Ah + Bk + C} \qquad \text{---(1)}$$

Choose $ah + bk + c = 0$ and $Ah + Bk + C = 0$

Solve the above equations for h and k.
Therefore, the differential equation (1) becomes,

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$$

which is a homogeneous equation.

Worked Examples

1. *Solve* $\frac{dy}{dx} = \frac{x + y + 3}{2x + 2y + 1}$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y + 3}{2x + 2y + 1} = \frac{x + y + 3}{2(x + y) + 1} \qquad \text{---(1)}$$

Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

(1) becomes, $\frac{dt}{dx} = \frac{t + 3}{2t + 1} + 1 = \frac{t + 3 + 2t + 1}{2t + 1}$

$$\frac{dt}{dx} = \frac{3t + 4}{2t + 1}$$

$$\frac{2t + 1}{3t + 4} dt = dx$$

$$\int \frac{2t + 1}{3t + 4} dt = \int dx + c \qquad \text{---(2)}$$

$3t + 4)2t + 1(2/3$
$\quad 2t + (8/3)$
$\quad \overline{\quad -(5/3)}$

$$\therefore \frac{2t + 1}{3t + 4} = \frac{2}{3} + \frac{-5/3}{3t + 4} = \frac{2}{3} - \frac{5}{3(3t + 4)} \qquad \text{---(3)}$$

Substituting equation (3) in (2), we get

$$\int\left[\frac{2}{3}-\frac{5}{3(3t+4)}\right]dt=\int dx+c$$

$$\frac{2}{3}t-\frac{5}{3}\frac{\log(3t+4)}{3}=x+c$$

$$\frac{2}{3}t-\frac{5}{9}\log(3t+4)=x+c$$

$$6t-5\log(3t+4)=9x+9c$$

$$6(x+y)-5\log[3(x+y)+4]-9x=c_1,\text{ where } c_1=9c$$

$$6y-3x-5\log(3x+3y+4)=c_1$$

2. *Solve* $\frac{dy}{dx}=\frac{4x-6y+1}{2x-3y+2}$

$$\Rightarrow \frac{dy}{dx}=\frac{4x-6y+1}{2x-3y+2}=\frac{2(2x-3y)+1}{2x-3y+2} \quad \text{---(1)}$$

Put $2x-3y=t \Rightarrow 2-3\frac{dy}{dx}=\frac{dt}{dx}$

$$2-\frac{dt}{dx}=3\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{1}{3}\left(2-\frac{dt}{dx}\right)$$

(1) becomes, $\frac{1}{3}\left(2-\frac{dt}{dx}\right)=\frac{2t+1}{t+2}$

$$2-\frac{dt}{dx}=\frac{3(2t-1)}{t+2}$$

$$\frac{dt}{dx}=2-\frac{3(2t+1)}{t+2}$$

$$\frac{dt}{dx}=\frac{2t+4-6t-3}{t+2}$$

$$\frac{dt}{dx}=\frac{-4t+1}{t+2}$$