

3. Find the angle between the pairs of curves

$$r^2 \sin 2\theta = 4 \qquad r^2 = 16 \sin 2\theta$$

$$\Rightarrow r^2 \sin 2\theta = 4 \qquad r^2 = 16 \sin 2\theta$$

Differentiating both equations with respect to θ , we get

$$r^2 \cos 2\theta \cdot 2 + \sin 2\theta \cdot 2r \frac{dr}{d\theta} = 0 \qquad 2r \frac{dr}{d\theta} = 16 \cos 2\theta \cdot 2$$

$$2 \sin 2\theta \cdot r \frac{dr}{d\theta} = -r^2 \cos 2\theta \cdot 2 \qquad r \frac{dr}{d\theta} = 16 \cos 2\theta$$

$$\frac{dr}{d\theta} = -\frac{r \cos 2\theta}{\sin 2\theta} \qquad \frac{dr}{d\theta} = \frac{16 \cos 2\theta}{r}$$

$$\frac{d\theta}{dr} = -\frac{\sin 2\theta}{r \cos 2\theta} \qquad \frac{d\theta}{dr} = \frac{r}{16 \cos 2\theta}$$

$$r \frac{d\theta}{dr} = \frac{\sin 2\theta}{\cos 2\theta} \qquad r \frac{d\theta}{dr} = \frac{r^2}{16 \cos 2\theta}$$

$$= -\tan 2\theta \qquad = \frac{16 \sin 2\theta}{16 \cos 2\theta}$$

$$= \tan(\pi - 2\theta) \qquad = \tan 2\theta$$

Therefore, $\tan \phi_1 = r(d\theta/dr) = \tan(\pi - 2\theta) \Rightarrow \phi_1 = \pi - 2\theta$

and $\tan \phi_2 = r(d\theta/dr) = \tan(2\theta) \Rightarrow \phi_2 = 2\theta$

$$\phi = |\phi_1 - \phi_2| = |\pi - 2\theta - 2\theta| = |\pi - 4\theta|$$

we have, $r^2 \sin 2\theta = 4 \qquad r^2 = 16 \sin 2\theta$

$$\frac{4}{\sin 2\theta} = 16 \sin 2\theta$$

$$(1/4) = \sin^2 2\theta$$

$$\sin 2\theta = 1/2$$

$$2\theta = \sin^{-1}(1/2) = \pi/6 \Rightarrow \theta = \pi/12$$

$$\therefore \phi = \left| \pi - \frac{4\pi}{12} \right| = \frac{2\pi}{3}$$

■

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4. Find the angle between the pairs of curves

$$r = a(1 - \cos \theta) \quad r = 2a \cos \theta$$

$$\rightarrow r = a(1 - \cos \theta)$$

$$\frac{dr}{d\theta} = a \sin \theta$$

$$\frac{d\theta}{dr} = \frac{1}{a \sin \theta}$$

$$r \frac{d\theta}{dr} = \frac{r}{a \sin \theta}$$

$$= \frac{a(1 - \cos \theta)}{a \sin \theta}$$

$$= \frac{2 \sin^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

$$r = 2a \cos \theta$$

$$\frac{dr}{d\theta} = -2a \sin \theta$$

$$\frac{d\theta}{dr} = -\frac{1}{2a \sin \theta}$$

$$r \frac{d\theta}{dr} = -\frac{r}{2a \sin \theta}$$

$$= -\frac{2a \cos \theta}{2a \sin \theta}$$

$$= -\cot \theta$$

$$= \tan\left(\frac{\pi}{2} + \theta\right)$$

Therefore, $\tan \phi_1 = r(d\theta/dr) = \tan(\theta/2) \Rightarrow \phi_1 = \theta/2$

and $\tan \phi_2 = r(d\theta/dr) = \tan((\pi/2) + \theta) \Rightarrow \phi_2 = (\pi/2) + \theta$

We have, $r = a(1 - \cos \theta)$

$r = 2a \cos \theta$

$$a(1 - \cos \theta) = 2a \cos \theta$$

$$1 - \cos \theta = 2 \cos \theta \Rightarrow 1 = 3 \cos \theta \Rightarrow \cos \theta = 1/3$$

$$\theta = \cos^{-1}(1/3)$$

$$\phi = |\phi_2 - \phi_1| = \left| \left(\frac{\pi}{2}\right) + \theta - \frac{\theta}{2} \right| = \frac{\pi}{2} + \frac{\theta}{2} = \frac{\pi}{2} + \frac{1}{2} \cos^{-1} \frac{1}{3}$$

$$\phi = \frac{1}{2} \left(\pi + \cos^{-1} \frac{1}{3} \right)$$

5. Find the angle between the pairs of curves

$$r = a \log \theta \quad r = \frac{a}{\log \theta}$$

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$$\begin{aligned} \Rightarrow r = a \log \theta & \quad r = \frac{a}{\log \theta} \\ \frac{dr}{d\theta} = \frac{a}{\theta} & \quad \frac{dr}{d\theta} = -\frac{a}{(\log \theta)^2} \cdot \frac{1}{\theta} \\ \frac{d\theta}{dr} = \frac{\theta}{a} & \quad \frac{d\theta}{dr} = -\frac{\theta(\log \theta)^2}{a} \\ r \frac{d\theta}{dr} = r \cdot \frac{\theta}{a} & \quad r \frac{d\theta}{dr} = -\frac{r\theta(\log \theta)^2}{a} \\ r \frac{d\theta}{dr} = \frac{a \log \theta \cdot \theta}{a} & \quad = -\frac{a \cdot \theta(\log \theta)^2}{\log \theta \cdot a} \\ \tan \phi_1 = r(d\theta/dr) = \theta \log \theta & \quad \tan \phi_2 = r(d\theta/dr) = -\theta \log \theta \end{aligned}$$

We have, $r = a \log \theta$ $r = \frac{a}{\log \theta}$
 $a \log \theta = \frac{a}{\log \theta}$ or $\log \theta = 1 \Rightarrow \theta = e$

$$\tan(\phi_1 - \phi_2) = \left| \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2} \right| = \left| \frac{2\theta \log \theta}{1 - \theta^2 (\log \theta)^2} \right| = \left| \frac{2e}{1 - e^2} \right|$$

$$\phi_1 - \phi_2 = \tan^{-1} \left| \frac{2e}{1 - e^2} \right| \quad \blacksquare$$

6. Show that the following pair of curves intersect orthogonally

$$\begin{aligned} r = a\theta & \quad r\theta = a \\ \Rightarrow r = a\theta & \quad r\theta = a \end{aligned}$$

Differentiating with respect to θ , we get

$$\begin{aligned} \frac{dr}{d\theta} = a & \quad r(1) + \theta \frac{dr}{d\theta} = 0 \\ \frac{d\theta}{dr} = \frac{1}{a} & \quad \theta \frac{dr}{d\theta} = -r \\ r \frac{d\theta}{dr} = \frac{r}{a} & \quad \frac{dr}{d\theta} = -\frac{r}{\theta} \end{aligned}$$

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$$\frac{d\theta}{dr} = -\frac{\theta}{r}$$

$$r \frac{d\theta}{dr} = -\theta = -\frac{a}{r}$$

$$\tan \phi_1 = r \left(\frac{d\theta}{dr} \right) = \frac{r}{a} \quad \text{---(1)}$$

$$\tan \phi_2 = r \left(\frac{d\theta}{dr} \right) = -\theta = -\frac{a}{r} \quad \text{---(2)}$$

From equations (1) and (2), we have

$$\tan \phi_1 \cdot \tan \phi_2 = \frac{r}{a} \left(-\frac{a}{r} \right) = -1$$

Hence, the given curves intersect orthogonally. ■

7. Show that the following pair of curves intersect orthogonally

$$r = 2 \sin \theta, \quad r = 2 \cos \theta$$

$$\Rightarrow r = 2 \sin \theta$$

$$\frac{dr}{d\theta} = 2 \cos \theta$$

$$\frac{d\theta}{dr} = \frac{r}{2 \cos \theta}$$

$$= \frac{2 \sin \theta}{2 \cos \theta}$$

$$= \tan \theta$$

$$\tan \phi_1 = \tan \theta \quad \text{---(1)}$$

$$\tan \phi_2 = -\cot \theta \quad \text{---(2)}$$

From equations (1) and (2), we have

$$\tan \phi_1 \cdot \tan \phi_2 = (\tan \theta)(-\cot \theta) = -1$$

Hence, the given curves intersect orthogonally. ■

8. Show that the following pair of curves intersect orthogonally

$$r^n = a^n \cos n\theta, \quad r^n = b^n \sin n\theta$$

$$\Rightarrow r^n = a^n \cos n\theta, \quad r^n = b^n \sin n\theta$$

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Differentiating with respect to θ , we get

$$nr^{n-1} \frac{dr}{d\theta} = a^n [-\sin n\theta \cdot n] \qquad nr^{n-1} \frac{dr}{d\theta} = b^n \cos n\theta \cdot n$$

$$r^{n-1} \frac{dr}{d\theta} = -a^n \sin n\theta \qquad r^{n-1} \frac{dr}{d\theta} = b^n \cos n\theta$$

$$\frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^{n-1}} \qquad \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{r^{n-1}}$$

$$\frac{d\theta}{dr} = -\frac{r^{n-1}}{a^n \sin n\theta} \qquad \frac{d\theta}{dr} = \frac{r^{n-1}}{b^n \cos n\theta}$$

$$r \frac{d\theta}{dr} = -\frac{r \cdot r^{n-1}}{a^n \sin n\theta} \qquad r \frac{d\theta}{dr} = \frac{r \cdot r^{n-1}}{b^n \cos n\theta}$$

$$\frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n} \qquad \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{r^n}$$

$$\tan \phi_1 = -\cot n\theta \qquad \tan \phi_2 = \tan n\theta$$

$$\tan \phi_1 = -\cot n\theta \qquad \tan \phi_2 = \tan n\theta$$

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$$\tan \phi_1 = -\cot n\theta \qquad \tan \phi_2 = \tan n\theta$$

$$\tan \phi_1 = -\cot n\theta \qquad \tan \phi_2 = \tan n\theta$$

From equations (1) and (2), we have

$$\tan \phi_1 \cdot \tan \phi_2 = (-\cot n\theta)(\tan n\theta) = -1$$

Hence, the given curves intersect orthogonally. ■

9. Show that the following pair of curves intersect orthogonally

$$r = a(1 + \cos \theta) \qquad r = b(1 - \cos \theta)$$

$$\Rightarrow r = a(1 + \cos \theta) \qquad r = b(1 - \cos \theta)$$

Differentiating with respect to θ , we get

$$\frac{dr}{d\theta} = a(-\sin \theta) \qquad \frac{dr}{d\theta} = b(0 - (-\sin \theta)) = b \sin \theta$$

$$\frac{d\theta}{dr} = \frac{1}{a \sin \theta} \qquad \frac{d\theta}{dr} = \frac{1}{b \sin \theta}$$

$$r \frac{d\theta}{dr} = -\frac{r}{a \sin \theta} \qquad r \frac{d\theta}{dr} = \frac{r}{b \sin \theta}$$

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$$\tan \phi_1 = -\frac{a(1 + \cos \theta)}{a \sin \theta} \quad \tan \phi_2 = \frac{b(1 - \cos \theta)}{b \sin \theta}$$

$$\tan \phi_1 = -\frac{(1 + \cos \theta)}{\sin \theta} \quad \tan \phi_2 = \frac{1 - \cos \theta}{\sin \theta} \quad \text{---(1) \quad ---(2)}$$

From equations (1) and (2), we have

$$\tan \phi_1 \cdot \tan \phi_2 = \left[-\frac{(1 + \cos \theta)}{\sin \theta} \right] \left[\frac{1 - \cos \theta}{\sin \theta} \right] = -\frac{(1 - \cos^2 \theta)}{\sin^2 \theta} = -\frac{\sin^2 \theta}{\sin^2 \theta}$$

$$\tan \phi_1 \cdot \tan \phi_2 = -1$$

Hence, the given curves intersect orthogonally. ■

10. Show that the following pair of curves intersect orthogonally

$$r = ae^\theta \quad re^\theta = a$$

$$\Rightarrow r = ae^\theta \quad re^\theta = a$$

Differentiating with respect to θ , we get

$$\frac{dr}{d\theta} = ae^\theta \quad re^\theta + \frac{dr}{d\theta} e^\theta = 0$$

$$\frac{dr}{d\theta} e^\theta = -re^\theta$$

$$\frac{d\theta}{dr} = \frac{1}{ae^\theta} \quad \frac{dr}{d\theta} = -r$$

$$r \frac{d\theta}{dr} = \frac{r}{ae^\theta} \quad \frac{d\theta}{dr} = \frac{1}{r}$$

$$\tan \phi_1 = \frac{ae^\theta}{ae^\theta} \quad r \frac{d\theta}{dr} = -1$$

$$\tan \phi_1 = 1 \quad \text{---(1)} \quad \tan \phi_2 = -1 \quad \text{---(2)}$$

From equations (1) and (2), we get

$$\tan \phi_1 \cdot \tan \phi_2 = (1)(-1) = -1$$

Hence, the given curves intersect orthogonally. ■

Exercises

I. Find the angle between the following pair of curves.

$$1) \quad r = \sin \theta + \cos \theta \quad r = 2 \sin \theta$$

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$$2) r = 6 \cos \theta \qquad r = 2(1 + \cos \theta)$$

$$3) r^n = a^n \sec(n\theta + \alpha) \qquad r^n = b^n \sec(n\theta + \beta)$$

$$4) r = \frac{a\theta}{1+\theta} \qquad r = \frac{a}{1+\theta^2}$$

$$5) r \sec^2\left(\frac{\theta}{2}\right) = a \qquad r \operatorname{cosec}^2\left(\frac{\theta}{2}\right) = b$$

$$6) r = a \cos \theta \qquad r = \frac{a}{2}$$

$$7) r^n = a^n (\sin n\theta + \cos n\theta) \qquad r^n = a^n \sin n\theta$$

$$8) r^2 \cos(2\theta + \alpha) = a^2 \qquad r^2 \cos(2\theta + \beta) = b^2$$

$$9) r^2 = a^2 \cos 2\theta + b^2 \qquad r = b$$

II Show that the following pairs of curves intersect orthogonally.

$$10) r^n \cos n\theta = a^n \qquad r^n \sin n\theta = b^n$$

$$11) \frac{2a}{r} = 1 - \cos \theta \qquad \frac{2a}{r} = 1 + \cos \theta$$

$$12) r^2 = a^2 \cos 2\theta \qquad r^2 = b^2 \sin 2\theta$$

$$13) r = a(1 + \sin \theta) \qquad r = a(1 - \sin \theta)$$

$$14) r^2 \sin 2\theta = a^2 \qquad r^2 \cos 2\theta = b^2$$

$$15) r = 16 \sec^2\left(\frac{\theta}{2}\right) \qquad r = 25 \operatorname{cosec}^2 \frac{\theta}{2}$$

Answers

I. 1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $|\alpha - \beta|$ 4) $\tan^{-1}(3)$

5) $\frac{\pi}{2}$ 6) $\frac{\pi}{3}$ 7) $\frac{\pi}{4}$ 8) $|\alpha - \beta|$

9) $\tan^{-1}\left(\frac{b^2}{a^2}\right)$

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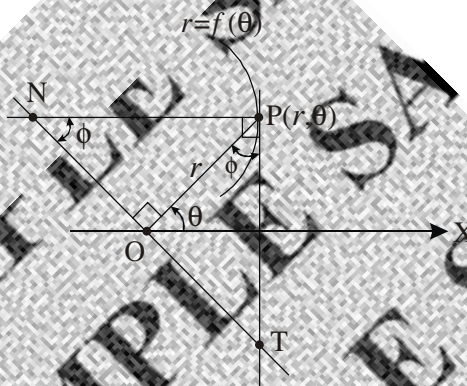
Polar Subtangent and Polar Subnormal

Figure 2.4

Let P be any point on the polar curve $r = f(\theta)$. Draw the tangent and the normal to the curve at P and a line perpendicular to the radius vector passing through the origin, which cuts the tangent at T and normal at N . Then length of OT is called *polar subtangent* and length of ON is called *polar subnormal*.

From right-angled triangle OPT , we have

$$\tan \phi = \frac{OT}{OP} = \frac{OT}{r}$$

$$OT = r \tan \phi = r \left(r \frac{d\theta}{dr} \right) = r^2 \frac{d\theta}{dr} = \text{Subtangent}$$

From right-angled triangle ONP , we have

$$\tan \phi = \frac{OP}{ON} = \frac{r}{ON}$$

$$ON = \frac{r}{\tan \phi} = \frac{r}{r(d\theta/dr)} = \frac{dr}{d\theta} = \text{Subnormal}$$

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Length of the Perpendicular from the Pole to Tangent

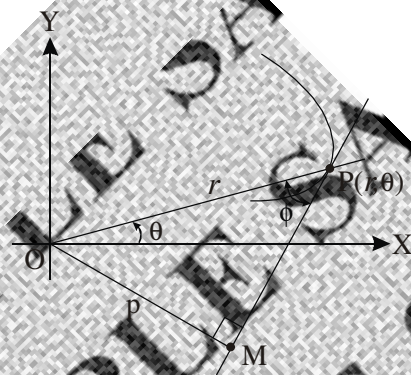


Figure 2.5

Let P be any point on the polar curve $r = f(\theta)$ draw a tangent to the curve at P . Draw a line perpendicular to the tangent from the origin (pole) to meet at M . Let $OM = p$, from the right-angled triangle OPM , we have

$$\sin \phi = \frac{OM}{OP} = \frac{p}{r}$$

$$p = r \sin \phi \quad \text{---(1)}$$

$$p^2 = r^2 \sin^2 \phi$$

$$\frac{1}{p^2} = \frac{1}{r^2 \sin^2 \phi} = \frac{1}{r^2} (\operatorname{cosec}^2 \phi) = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$= \frac{1}{r^2} \left(1 + \frac{1}{\tan^2 \phi} \right) \quad \left[\tan \phi = r \frac{d\theta}{dr} \right]$$

$$= \frac{1}{r^2} \left(1 + \frac{1}{r^2 (d\theta/dr)^2} \right) = \frac{1}{r^2} \left(1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \quad \text{---(2)}$$

Note If $\frac{1}{r} = u$, then $-\frac{1}{r^2} \frac{dr}{d\theta} = \frac{du}{d\theta}$

Squaring on both the sides, we get

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$$\left(-\frac{1}{r^2} \frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2$$

$$\frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2$$

$$\therefore \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Pedal Equation (p-r equation)

The relation connecting between p and r is called *pedal* or *p-r equation*.

To find the pedal equation we use the following formulae.

$$\text{i) } p = r \sin \phi$$

$$\text{ii) } \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2$$

Worked Examples

1. Find the polar subtangent and the polar subnormal to the curve

$$r = a(1 - \cos \theta)$$

$$\rightarrow r = a(1 - \cos \theta)$$

$$\frac{dr}{d\theta} = a \sin \theta$$

$$\frac{d\theta}{dr} = \frac{1}{a \sin \theta}$$

$$r^2 \frac{d\theta}{dr} = \frac{r^2}{a \sin \theta}$$

$$= \frac{a^2(1 - \cos \theta)^2}{a \sin \theta} = \frac{a(1 - \cos \theta)^2}{\sin \theta}$$

2. Find the polar subtangent and the polar subnormal to the curve

$$r = e^{\theta \cot \alpha}$$

$$\rightarrow r = e^{\theta \cot \alpha}$$

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$$\frac{dr}{d\theta} = e^{\theta \cot \alpha} \cdot \cot \alpha = \text{polar subnormal.}$$

$$\frac{d\theta}{dr} = \frac{1}{e^{\theta \cot \alpha} \cot \alpha} = \frac{1}{r \cot \alpha}$$

$$r^2 \frac{d\theta}{dr} = \frac{r^2}{r \cot \alpha} = \frac{r}{\cot \alpha} = \frac{e^{\theta \cot \alpha}}{\cot \alpha} = \text{polar subtangent.} \quad \blacksquare$$

3. Find the pedal equation to the curve $r = a(1 - \cos \theta)$

$$\rightarrow r = a(1 - \cos \theta)$$

$$\frac{dr}{d\theta} = a \sin \theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} (a \sin \theta)^2 = \frac{1}{r^2} + \frac{1}{r^4} a^2 \sin^2 \theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} (1 - \cos^2 \theta) \quad \text{---(1)}$$

$$\text{Given, } r = a(1 - \cos \theta) \Rightarrow \frac{r}{a} = 1 - \cos \theta$$

$$\cos \theta = 1 - \frac{r}{a} = \frac{a - r}{a} \quad \text{---(2)}$$

Substituting equation (2) in (1), we get

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} \left[1 - \left(\frac{a - r}{a} \right)^2 \right]$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4} \left[1 - \frac{(a - r)^2}{a^2} \right]$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4} \left[\frac{a^2 - (a - r)^2}{a^2} \right]$$

$$= \frac{1}{r^2} + \frac{a^2 - (a^2 + r^2 - 2ar)}{r^4}$$

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$$= \frac{1}{r^2} + \frac{2ar - r^2}{r^4} = \frac{1}{r^2} + \frac{2ar}{r^4} - \frac{r^2}{r^4}$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

$$r^3 = 2ap^2 \quad \blacksquare$$

4. Find the pedal equation to the curve $2a = r(1 - \cos \theta)$

$$\rightarrow 2a = r(1 - \cos \theta)$$

$$\frac{2a}{r} = (1 - \cos \theta)$$

Differentiating with respect to θ , we get

$$\frac{-2a}{r^2} \cdot \frac{dr}{d\theta} = \sin \theta$$

$$\frac{dr}{d\theta} = \frac{r^2 \sin \theta}{2a}$$

We have,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{-r^2 \sin \theta}{2a} \right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} \frac{r^4 \sin^2 \theta}{4a^2} = \frac{1}{r^2} + \frac{\sin^2 \theta}{4a^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{4a^2} (1 - \cos^2 \theta) \quad \text{---(1)}$$

Given $2a = r(1 - \cos \theta)$

$$(1 - \cos \theta) = \frac{2a}{r}$$

$$\cos \theta = 1 - \frac{2a}{r}$$

$$\cos \theta = \frac{r - 2a}{r} \quad \text{---(2)}$$

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Substituting equating (1) and (2), we get

$$\begin{aligned}\frac{1}{p^2} &= \frac{1}{r^2} + \frac{1}{4a^2} \left[1 - \left(\frac{r-2a}{r} \right)^2 \right] \\ &= \frac{1}{r^2} + \frac{1}{4a^2} \left[1 - \frac{(r^2 - 4ar + 4a^2)}{r^2} \right] \\ &= \frac{1}{r^2} + \frac{1}{4a^2} \left[\frac{r^2 - r^2 + 4ar - 4a^2}{r^2} \right] \\ &= \frac{1}{r^2} + \frac{1}{4a^2} \left[\frac{4ar - 4a^2}{r^2} \right] = \frac{1}{r^2} + \frac{1}{4a^2} \frac{4a(r-a)}{r^2} \\ \frac{1}{p^2} &= \frac{1}{r^2} + \frac{r-a}{ar^2}\end{aligned}$$

5. Find the pedal equation to the curve $r = 2a(1 + \cos \theta)$

$$\rightarrow r = 2a(1 + \cos \theta)$$

$$\frac{dr}{d\theta} = 2a(-\sin \theta)$$

$$\frac{dr}{d\theta} = -2a \sin \theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} (-2a \sin \theta)^2 = \frac{1}{r^2} + \frac{1}{r^4} (4a^2 \sin^2 \theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{4a^2}{r^4} (1 - \cos^2 \theta) \quad \text{---(1)}$$

Given, $r = 2a(1 - \cos \theta)$

$$\frac{r}{2a} = 1 - \cos \theta$$

$$\cos \theta = 1 - \frac{r}{2a}$$

$$\cos \theta = \frac{2a - r}{2a} \quad \text{---(2)}$$

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Substituting equation (2) in (1), we get

$$\begin{aligned}\frac{1}{p^2} &= \frac{1}{r^2} + \frac{4a^2}{r^4} \left[1 - \left(\frac{2a-r}{2a} \right)^2 \right] \\ \frac{1}{p^2} &= \frac{1}{r^2} + \frac{4a^2}{r^4} \left[1 - \left(\frac{4a^2 - 4ar + r^2}{4a^2} \right) \right] \\ &= \frac{1}{r^2} + \frac{4a}{r^4} \left[\frac{4a^2 - 4a^2 + 4ar - r^2}{4a^2} \right] = \frac{1}{r^2} + \frac{1}{r^4} [4ar - r^2] \\ &= \frac{1}{r^2} + \frac{r(4a-r)}{r^4} = \frac{1}{r^2} + \frac{4a}{r^3} - \frac{1}{r^2} \\ \frac{1}{p^2} &= \frac{4a}{r^3} \\ r^3 &= 4ap^2\end{aligned}$$

6. Find the pedal equation to the curve $r^m = a^m \cos m\theta$

$$\rightarrow r^m = a^m \cos m\theta$$

$$mr^{m-1} \frac{dr}{d\theta} = a^m (-\sin m\theta \cdot m)$$

$$r^{m-1} \frac{dr}{d\theta} = -a^m \sin m\theta$$

$$\frac{dr}{d\theta} = -\frac{a^m \sin m\theta}{r^{m-1}}$$

We have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(-\frac{a^m \sin m\theta}{r^{m-1}} \right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} \frac{a^{2m} \sin^2 m\theta}{r^{2(m-1)}} = \frac{1}{r^2} + \frac{a^{2m} (1 - \cos^2 m\theta)}{r^{2m-2+4}}$$

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$$\begin{aligned}
 &= \frac{1}{r^2} + \frac{a^{2m}}{r^{2m+2}} \left[1 - \left(\frac{r^m}{a^m} \right)^2 \right] = \frac{1}{r^2} + \frac{a^{2m}}{r^{2m+2}} \left[\frac{a^{2m} - r^{2m}}{a^{2m}} \right] \\
 &= \frac{1}{r^2} + \frac{a^{2m} - r^{2m}}{r^{2m+2}} = \frac{1}{r^2} + \frac{a^{2m}}{r^{2m+2}} - \frac{r^{2m}}{r^{2m+2}} \\
 &= \frac{1}{r^2} + \frac{a^{2m}}{r^{2m+2}} - \frac{1}{r^2} \\
 &\frac{1}{p^2} = \frac{a^{2m}}{r^{2m+2}} \\
 &\quad r^{2m+2} = a^{2m} p^2
 \end{aligned}$$

7. Find the pedal equation to the curve, $\frac{l}{r} = 1 + e \cos \theta$

► $\frac{l}{r} = 1 + e \cos \theta$

Differentiating with respect to θ , we get

$$\begin{aligned}
 -\frac{d}{dr} \frac{dr}{d\theta} &= e(-\sin \theta) \\
 \frac{dr}{d\theta} &= \frac{er^2 \sin \theta}{l}
 \end{aligned}$$

We have,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{er^2 \sin \theta}{l} \right)^2 = \frac{1}{r^2} + \frac{1}{r^4} \frac{e^2 r^4 \sin^2 \theta}{l^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} (1 - \cos^2 \theta) \quad \dots(1)$$

Given,

$$\frac{l}{r} = 1 + e \cos \theta$$

$$e \cos \theta = \left(\frac{l}{r} - 1 \right)$$

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$$\cos \theta = \frac{1}{e} \left(\frac{l}{r} - 1 \right) = \frac{1}{e} \left(\frac{l-r}{r} \right)$$

$$\cos \theta = \frac{l-r}{er} \quad \text{---(2)}$$

Substituting equation (1) and (2), we get

$$\begin{aligned} \frac{1}{p^2} &= \frac{1}{r^2} + \frac{e^2}{l^2} \left[1 - \left(\frac{l-r}{er} \right)^2 \right] \\ &= \frac{1}{r^2} + \frac{e^2}{l^2} \left[1 - \frac{(l^2 + r^2 - 2lr)}{e^2 r^2} \right] \\ &= \frac{1}{r^2} + \frac{e^2}{l^2} \left[\frac{e^2 r^2 - l^2 - r^2 + 2lr}{e^2 r^2} \right] \\ &= \frac{1}{r^2} + \frac{e^2 r^2 - l^2 - r^2 + 2lr}{l^2 r^2} \\ &= \frac{1}{r^2} + \frac{e^2 r^2 - r^2 + 2lr}{l^2 r^2} - \frac{l^2}{l^2 r^2} \\ \frac{1}{p^2} &= \frac{e^2 r^2 - r^2 + 2lr}{l^2 r^2} \quad \blacksquare \end{aligned}$$

8. Find the pedal equation to the curve $r = e^{\theta \cot \alpha}$

$$\rightarrow r = e^{\theta \cot \alpha}$$

$$\frac{dr}{d\theta} = e^{\theta \cot \alpha} \cot \alpha$$

$$\frac{dr}{d\theta} = r \cot \alpha$$

We have,

$$\begin{aligned} \frac{1}{p^2} &= \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \\ &= \frac{1}{r^2} + \frac{1}{r^4} (r \cot \alpha)^2 = \frac{1}{r^2} + \frac{r^2 \cot^2 \alpha}{r^4} \end{aligned}$$

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$$= \frac{1}{r^2} + \frac{\cot^2 \alpha}{r^2} = \frac{1 + \cot^2 \alpha}{r^2}$$

$$\frac{1}{p^2} = \frac{\operatorname{cosec}^2 \alpha}{r^2}$$

$$r^2 = p^2 \operatorname{cosec}^2 \alpha$$

9. Find the pedal equation to the curve $r = a(1 + \sin \theta)$

$$\rightarrow r = a(1 + \sin \theta)$$

$$\frac{dr}{d\theta} = a \cos \theta$$

We have,
$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} (a \cos \theta)^2$$

$$= \frac{1}{r^2} + \frac{a^2 \cos^2 \theta}{r^4}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} (1 - \sin^2 \theta) \quad \text{---(1)}$$

Given that $r = a(1 + \sin \theta)$

$$\frac{r}{a} = 1 + \sin \theta$$

$$\sin \theta = \frac{r}{a} - 1 = \frac{r - a}{a} \quad \text{---(2)}$$

Substituting (2) in (1), we get

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} \left[1 - \left(\frac{r - a}{a} \right)^2 \right]$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4} \left[1 - \frac{(r - a)^2}{a^2} \right]$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4} \left[\frac{a^2 - [r^2 + a^2 - 2ar]}{a^2} \right]$$

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$$\begin{aligned}
 &= \frac{1}{r^2} + \frac{a^2 - r^2 - a^2 + 2ar}{r^4} \\
 &= \frac{1}{r^2} + \frac{2ar - r^2}{r^4} = \frac{1}{r^2} + \frac{2ar}{r^4} - \frac{r^2}{r^4} \\
 \frac{1}{p^2} &= \frac{2a}{r^3} \\
 2a p^2 &= r^3 \quad \blacksquare
 \end{aligned}$$

Exercises

I. Find the polar subtangent for the following curves.

$$1) \frac{2a}{r} = 1 + e \cos \theta \qquad 2) r = a(1 - \cos \theta)$$

II. Find the polar sub normal for the following curves.

$$1) r = \frac{a}{\theta} \qquad 2) r = a(1 + \cos \theta)$$

III. Find the pedal equation for the following curves.

$$1) r = a(1 + \cos \theta) \qquad 2) \frac{2a}{r} = 1 - \cos \theta$$

$$3) r^2 = a^2 \sec 2\theta \qquad 4) r = a\theta$$

$$5) r = a \cos e c^2 \frac{\theta}{2} \qquad 6) r = a \sin m\theta$$

$$7) r^2 = a^2 \sin 2\theta \qquad 8) r^m \cos m\theta = a^m$$

$$9) r = a + b \cos \theta \qquad 10) r^2 = a^2 \sin^2 \theta$$

$$11) r^n = a(1 + \cos n\theta) \qquad 12) r = a(1 + \sin \theta)$$

$$13) r^m = a^m \sin m\theta + b^m \cos m\theta$$

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Answers

- I. 1) $\frac{2a}{e} \cos \theta$ 2) $2a \cot \frac{\theta}{2} \cos^2 \frac{\theta}{2}$
- II. 1) $-\frac{a}{\theta^2}$ 2) $a \sin \theta$
- III. 1) $2ap^2 = r^2$ 2) $p^2 = ar$
 3) $pr = a^2$ 4) $p^2(r^2 + a^2) = r^4$
 5) $p^2 = ar$ 6) $r^4 = p^2[a^2m^2 + (1-m^2)r^2]$
 7) $r^3 = a^2p$ 8) $pr^{m-1} = a^m$
 9) $r^4 = [(b^2 - a^2) + 2ar] p^2$ 10) $r^3 = a^2p$
 11) $r^{n+1} = 2ap^2$ 12) $r^3 = 2ap^2$
 13) $r^{m+1} = p\sqrt{a^{2m} + b^{2m}}$

Taylor’s Theorem for a Function of Single Variable

Let $f(a+x)$ be a function of x which can be expanded in powers of x and let the expansion be differentiable any number of times with respect to x , then

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \dots + \frac{x^n}{n!} f^{(n)}(a) + \dots$$

Proof Let $f(a+x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \dots$, ---(1)

where A_0, A_1, A_3, \dots are constants.

By successive differentiation with respect to x , we get

$$f'(a+x) = A_1 + 2A_2x + 3A_3x^2 + \dots,$$

$$f''(a+x) = 2.1.A_2 + 3.2.A_3x + 4.3.A_4x^2 + \dots,$$

$$f'''(a+x) = 3.2.1.A_3 + 4.3.2.A_4x + \dots,$$

Putting $x = 0$ in each of these, we get

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$$f(a) = A_0, \quad f'(a) = A_1,$$

$$f''(a) = 2!A_2 \Rightarrow A_2 = \frac{f''(a)}{2!},$$

$$f'''(a) = 3!A_3 \Rightarrow A_3 = \frac{f'''(a)}{3!},$$

Substituting these values in (1), we get

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \dots$$

$$+ \frac{x^n}{n!}f^{(n)}(a) + \dots$$

Remarks

1) If $a = 0$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

which is the Maclaurin's series, (it is discussed afterwards)

2) If $x = h$, then the series in powers of h is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots$$

3) If $x = h - a$, then

$$f(h) = f(a) + (h-a)f'(a) + \frac{(h-a)^2}{2!}f''(a) + \dots$$

$$4) f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

Maclaurin's series

Let $f(x)$ be a function of x which can be expanded in powers of x and let the expansion be differentiable term by term any number of times, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

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