(iii) $x = \text{cis}\,\alpha \Rightarrow x^n = \text{cis}\,n\alpha$

$y = \text{cis}\,\beta \Rightarrow y^m = \text{cis}\,m\beta$

$$\frac{x^n}{y^m} + \frac{y^m}{x^n} = \frac{\text{cis}\,n\alpha}{\text{cis}\,m\beta} + \frac{\text{cis}\,m\beta}{\text{cis}\,n\alpha}$$

$$= \text{cis}(n\alpha - m\beta) + \frac{1}{\text{cis}\,(n\alpha - m\beta)}$$

$$= \cos(n\alpha - m\beta) + i\sin(n\alpha - m\beta)$$
$$+ \cos(n\alpha - m\beta) - i\sin(n\alpha - m\beta)$$
$$= 2\cos(n\alpha - m\beta)$$

(iv) $$\frac{x^n y^m}{z^l} + \frac{z^l}{x^n y^m} = \frac{\text{cis}\,n\alpha\,\text{cis}\,m\beta}{\text{cis}\,l\gamma} + \frac{\text{cis}\,l\gamma}{\text{cis}\,n\alpha\,\text{cis}\,m\beta}$$

$$= \text{cis}\,(n\alpha + m\beta - l\gamma) + \frac{1}{\text{cis}\,(n\alpha + m\beta - l\gamma)}$$

$$= \cos(n\alpha + m\beta - l\gamma) + i\sin(n\alpha + m\beta - l\gamma)$$
$$+ \cos(n\alpha + m\beta - l\gamma) - i\sin(n\alpha + m\beta - l\gamma)$$
$$= 2\cos(n\alpha + m\beta - l\gamma)$$ ■

7. *If* $\cos\alpha + \cos\beta + \cos\gamma = 0$ *and* $\sin\alpha + \sin\beta + \sin\gamma = 0$, *then prove the following*

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(ii) $\cos 2\alpha + \cos 2\gamma + \cos 2\gamma = 0$

$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(iii) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$

$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$

(iv) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{3}{2}$

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{3}{2}$

➡ (i) $\cos\alpha + \cos\beta + \cos\gamma = 0$ ---(1)

$\sin\alpha + \sin\beta + \sin\gamma = 0$ ---(2)

Take equation (1) + i (2), we get

$$(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$$

$$\text{cis}\,\alpha + \text{cis}\,\beta + \text{cis}\,\gamma = 0$$

Put $x = \text{cis}\,\alpha,\ \ y = \text{cis}\,\beta$ and $z = \text{cis}\,\gamma$

$$\therefore\ x + y + z = 0$$

$$\Rightarrow x^3 + y^3 + z^3 = 3xyz$$

i.e., $(\text{cis}\,\alpha)^3 + (\text{cis}\,\beta)^3 + (\text{cis}\,\gamma)^3 = 3\,\text{cis}\,\alpha\ \text{cis}\,\beta\ \text{cis}\,\gamma$

$$\text{cis}\,3\alpha + \text{cis}\,3\beta + \text{cis}\,3\gamma = 3\,\text{cis}\,(\alpha+\beta+\gamma)$$

$$\cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$$

$$= 3(\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma))$$

$$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$$

$$= 3\cos(\alpha+\beta+\gamma) + 3i\sin(\alpha+\beta+\gamma)$$

$$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$$

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$$

(ii) Let $x = \text{cis}\,\alpha\quad y = \text{cis}\,\beta\quad z = \text{cis}\,\gamma$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{\text{cis}\,\alpha} + \frac{1}{\text{cis}\,\beta} + \frac{1}{\text{cis}\,\gamma}$$

$$= \text{cis}(-\alpha) + \text{cis}(-\beta) + \text{cis}(-\gamma)$$

$$= (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$$

$$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= 0 + i0 = 0$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \qquad \text{---(3)}$$

Since $x + y + z = 0$

$$\Rightarrow (x+y+z)^2 = 0$$

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 0$$

$$x^2 + y^2 + z^2 + 2xyz\left(\frac{1}{z} + \frac{1}{x} + \frac{1}{y}\right) = 0$$

$$x^2 + y^2 + z^2 + 2xyz(0) = 0$$

$$(\text{cis}\,\alpha)^2 + (\text{cis}\,\beta)^2 + (\text{cis}\,\gamma)^2 = 0$$
$$\text{cis}\,2\alpha + \text{cis}\,2\beta + \text{cis}\,2\gamma = 0$$
$$(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$$
$$\therefore \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$
$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

(iii) From equation (3), we have

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$
$$\frac{yz + zx + xy}{xyz} = 0$$
$$xy + yz + xz = 0$$
$$\text{cis}\,\alpha\ \text{cis}\,\beta + \text{cis}\,\beta\,\text{cis}\,\gamma + \text{cis}\,\alpha\,\text{cis}\,\gamma = 0$$
$$\text{cis}(\alpha+\beta) + \text{cis}(\beta+\gamma) + \text{cis}(\alpha+\gamma) = 0$$
$$\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\alpha+\gamma)$$
$$+\, i(\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\alpha+\gamma)) = 0$$
$$\therefore \cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\alpha+\gamma) = 0$$
$$\sin(\alpha+\beta)\sin(\beta+\gamma) + \sin(\alpha+\gamma) = 0$$

(iv) From (ii), we have

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$
$$(2\cos^2\alpha - 1) + (2\cos^2\beta - 1) + (2\cos^2\gamma - 1) = 0$$
$$2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 = 0$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2$$
$$(1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 3/2$$
$$3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 3/2$$
$$3 - \frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$$

Therefore, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$ ■

8. *Prove that* $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n=\cos n\theta+i\sin n\theta$

➡ Let $z=\cos\theta+i\sin\theta$ and $\frac{1}{z}=\cos\theta-i\sin\theta$

Therefore, $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n=\left(\frac{1+z}{1+1/z}\right)^n$

$$=\left(\frac{1+z}{(z+1)/z}\right)^n=\left(\frac{1}{1/z}\right)^n=z^n$$

$$=(\cos\theta+i\sin\theta)^n$$

$$=\cos n\theta+i\sin n\theta \qquad \blacksquare$$

9. *If n is a positive integer, then show that*

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)$$

➡ $$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)=\left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)}\right]^n \quad \text{---(1)}$$

Let $z=\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)$

Then $\frac{1}{z}=\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)$

Therefore, equation (1) becomes,

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)=\left(\frac{1+z}{1+\frac{1}{z}}\right)^n$$

$$= \left(\frac{1+z}{(z+1)/z}\right)^n = \left(\frac{1}{1/z}\right)^n = z^n$$

$$= \left[\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$$

$$= \left[\cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)\right] \qquad ■$$

10. *If* $a+ib=\dfrac{1}{\alpha+i\beta}$, *prove that* $\left(\alpha^2+\beta^2\right)\left(a^2+b^2\right)=1$

➡ $a+ib=\dfrac{1}{\alpha+i\beta}\times\dfrac{\alpha-i\beta}{\alpha-i\beta}$

$$a+ib=\frac{\alpha-i\beta}{\alpha^2-i^2\beta^2}=\frac{\alpha-i\beta}{\alpha+i\beta^2}$$

$$\Rightarrow |a+ib|=\frac{1}{\alpha^2+\beta^2}|\alpha-i\beta|$$

$$\sqrt{a^2+b^2}=\frac{1}{\alpha^2+\beta^2}\sqrt{\alpha^2+(-\beta)^2}=\frac{1}{\alpha^2+\beta^2}\sqrt{\alpha^2+\beta^2}$$

$$\sqrt{a^2+b^2}=\frac{1}{\sqrt{\alpha^2+\beta^2}}$$

$$\sqrt{a^2+b^2}\sqrt{\alpha^2+\beta^2}=1$$

$$\left(a^2+b^2\right)\left(\alpha^2+\beta^2\right)=1 \qquad ■$$

11. *If* $\left(a_1+ib_1\right)\left(a_2+ib_2\right)---\left(a_n+ib_n\right)=A+iB$, *prove that*

(i) $\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)---\left(a_n^2+b_n^2\right)=A^2+B^2$

(ii) $\tan^{-1}\dfrac{b_1}{a_1}+\tan^{-1}\dfrac{b_2}{a_2}+---+\tan^{-1}\dfrac{b_n}{a_n}=\tan^{-1}\dfrac{B}{A}$

➡ Let $a_1+ib_1=r_1\operatorname{cis}\theta_1 \Rightarrow r_1=\sqrt{a_1^2+b_1^2}$ and $\theta_1=\tan^{-1}\dfrac{b_1}{a_1}$

$$a_2 + ib_2 = r_2\text{cis}\theta_2 \Rightarrow r = \sqrt{a_2^2 + b_2^2} \text{ and } \theta_2 = \tan^{-1}\frac{b_2}{a_2}$$

$$a_n + ib_n = r_n\text{cis}\theta_\text{n} \Rightarrow r_n = \sqrt{a_n^2 + b_n^2} \text{ and } \theta_n = \tan^{-1}\frac{b_n}{a_n}$$

and $$A + iB = r\text{cis}\theta \Rightarrow r = \sqrt{A^2 + B^2} \text{ and } \theta = \tan^{-1}\frac{B}{A}$$

Therefore,

$$(a_1 + ib_1)(a_2 + ib_2)---(a_n + ib_n) = A + iB$$

$$\Rightarrow r_1\text{cis}\theta_1 r_2\text{cis}\theta_2 --- r_n\text{cis}\theta_n = r\text{cis}\theta$$

$$\Rightarrow r_1 r_2 --- r_n\text{cis}(\theta_1 + \theta_2 + ---- + \theta_n) = r\text{cis}\theta$$

$$\Rightarrow r = r_1 r_2 --- r_n \text{ and } \theta = \theta_1 + \theta_2 + ---- + \theta_n$$

$$\Rightarrow \sqrt{A^2 + B^2} = \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2} --- \sqrt{a_n^2 + b_n^2}$$

$$\Rightarrow A^2 + B^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2)---(a_n^2 + b_n^2)$$

and $$\theta_1 + \theta_2 + ---- + \theta_n = \theta$$

$$\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b^2}{a^2} + ---- + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$$ ■

12\. *If* $x = \cos\theta + i\sin\theta$, *then* $\dfrac{x^{2n} - 1}{x^{2n} + 1} = i\tan n\theta$

➡ $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$x^n = \cos n\theta + i\sin n\theta$$

$$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Consider, $\dfrac{x+\dfrac{1}{x^n}}{x^n-\dfrac{1}{x^n}}=\dfrac{2i\sin n\theta}{2\cos n\theta}$

$$\frac{\dfrac{x^{2n}+1}{x^n}}{\dfrac{x^{2n}-1}{x^n}}=i\tan n\theta$$

$$\therefore \frac{x^{2n}+1}{x^{2n}-1}=i\tan n\theta$$

13. *Prove that following*

(i) $\cos x=\dfrac{e^{ix}+e^{-ix}}{2}$, (ii) $\sin x=\dfrac{e^{ix}-e^{-ix}}{2i}$

➡ We have $e^{ix}=\cos x+i\sin x$ ---(1)

$e^{-ix}=\cos x-i\sin x$ ---(2)

Adding equations (1) and (2), we get

$$e^{ix}+e^{-ix}=2\cos x\Rightarrow\frac{e^{ix}+e^{-ix}}{2}=\cos x$$

Subtracting equation (2) from (1), we get

$$e^{ix}-e^{-ix}=2i\sin x\Rightarrow\frac{e^{ix}-e^{-ix}}{2i}=\sin x$$ ■

14. *Prove the following*

(i) $\sin ix=i\sinh x$, (ii) $\cos ix=\cosh x$, (iii) $\tan ix=i\tanh x$

➡ (i) We have $\sin y=\dfrac{e^{iy}-e^{-iy}}{2i}$

Put $y=ix$

$$\sin ix=\frac{e^{i(ix)}-e^{-i(ix)}}{2i}$$

$$=\frac{e^{i^2x}-e^{-i^2x}}{2i}=\frac{e^{-x}-e^{x}}{2i}$$

$$= -i\left[\frac{e^{-x} - e^{x}}{2}\right] \qquad \because \frac{1}{i} = -i$$

$$= i\left(\frac{e^{x} - e^{-x}}{2}\right)$$

(ii) $\sin ix = i \sinh x$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

Put $y = ix$

$$\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x$$

(iii) $\tan ix = \dfrac{\sin ix}{\cos ix} = \dfrac{i \sinh x}{\cosh x} = i \tanh x$ ■

15. *Express the following in the form of* $a + ib$

(a) $\sin(x+iy)$ (b) $\cos(x+iy)$
(c) $\tan(x+iy)$ (d) $\cot(x+iy)$
(e) $\sec(x+iy)$ (f) $\operatorname{cosec}(x+iy)$

➡ (a) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$$= \sin x[\cosh y] + \cos x[i \sinh y]$$

$$= \sin x \cosh y + i \cos x \sinh y$$

(b) $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$

$$= \cos x \cosh y - \sin x (i \sinh y)$$

$$= \cos x \cosh y - i \sin x \sinh y$$

(c) $\tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)}$

$$= \frac{\sin(x+iy)\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\frac{1}{2}[\sin(x+iy+x-iy) + \sin(x+iy-(x-iy))]}{\frac{1}{2}[\cos[x+iy+(x-iy)] + \cos(x+iy-(x-iy))]}$$

$$=\frac{\sin 2x+\sin(2iy)}{\cos 2x+\cos(2iy)}$$

$$=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$

$$=\frac{\sin 2x}{\cos 2x+\cosh 2y}+i\frac{\sinh 2y}{\cos 2x+\cosh 2y}$$

(d) $$\cot(x+iy)=\frac{\cos(x+iy)}{\sin(x+iy)}\times\frac{\sin(x-iy)}{\sin(x-iy)}$$

$$=\frac{\frac{1}{2}[\sin(x+iy+x-iy)-\sin[x+iy-(x-iy)]]}{\frac{1}{2}[\cos[x+iy-(x-iy)]-\cos(x+iy+(x-iy))]}$$

$$=\frac{\sin 2x-\sin 2iy}{\cos 2iy-\cos 2x}$$

$$=\frac{\sin 2x-i\sinh 2y}{\cosh 2y-\cos 2x}$$

$$\cot(x+iy)=\frac{\sin 2x}{\cosh 2y-\cos 2x}-i\frac{\sinh 2y}{\cosh 2y-\cos 2x}$$

(e) $$\sec(x+iy)=\frac{1}{\cos(x+iy)}\times\frac{\cos(x-iy)}{\cos(x-iy)}$$

$$=\frac{\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$$

$$=\frac{\cos x\cos iy+\sin x\sin iy}{\frac{1}{2}[\cos(x+iy+x-iy)+\cos[x+iy-(x-iy)]]}$$

$$=\frac{2[\cos x\cosh y+i\sin x\sinh y]}{\cos 2x+\cos 2iy}$$

$$=\frac{2[\cos x\cosh y+i\sin x\sinh y]}{\cos 2x+\cosh 2y}$$

$$=\frac{2\cos x\cosh y}{\cos 2x+\cosh 2y}+i\frac{2\sin x\sinh y}{\cos 2x+\cosh 2y}$$

(f) $\operatorname{cosec}(x+iy)=\frac{1}{\sin(x+iy)}\frac{\sin(x-iy)}{\sin(x-iy)}$

$$=\frac{\sin x\cos iy-\cos x\sin iy}{\frac{1}{2}[\cos(x+iy-(x-iy))-\cos(x+iy+x-iy)]}$$

$$=\frac{2[\sin x\cosh y-\cos x(i\sinh y)]}{\cosh i2y-\cos 2x}$$

$$=\frac{2[\sin x\cosh y-i\cos x\sinh y]}{\cosh 2y-\cos 2x}$$

$$=\frac{2\sin x\cosh y}{\cosh 2y-\cos 2x}-i\frac{2\cos x\sinh y}{\cosh 2y-\cos 2x}$$

16. *Express the following in the form of* $a+ib$

(i) $\sinh(x+iy)$, (ii) $\cosh(x+iy)$, (iii) $\tanh(x+iy)$

➡ (i) $\sinh(x+iy)=\frac{i}{i}\sinh(x+iy)$

$$=\frac{1}{i}\sin i(x+iy)=-i\sin(ix-y)$$

$$=-i[\sin ix\cos y-\cos ix\sin y]$$

$$=-i[i\sinh x\cos y-\cosh x\sin y]$$

$$=\sinh x\cos y+i\cosh x\sin y$$

(ii) $\cosh(x+iy)=\cos i(x+iy)$

$$=\cos(ix-y)$$

$$=\cos ix\cos y+\sin ix\sin y$$

$$=\cosh x\cos y+i\sinh x\sin y$$

(iii) $\tanh(x+iy)=\frac{i}{i}\tanh(x+iy)$

$$=\frac{1}{i}\tan i(x+iy)$$

$$=-i\tan(ix-y)$$

$$= -\frac{i\sin(ix-y)}{\cos(ix-y)} \times \frac{\cos(ix+y)}{\cos(ix+y)}$$

$$= -i\frac{\frac{1}{2}[\sin(ix-y+ix+y)+\sin(ix-y-ix-y)]}{\frac{1}{2}[\cos(ix-y+ix+y)+\cos(ix-y-ix-y)]}$$

$$= \frac{-i[\sin 2ix+\sin(-2y)]}{\cos 2ix+\cos(-2y)}$$

$$= \frac{-i(i\sinh 2x-\sin 2y)}{\cosh 2x+\cos 2y} = \frac{\sinh 2x+i\sin 2y}{\cosh 2x+\cos 2y}$$

$$= \frac{\sinh 2x}{\cosh 2x+\cos 2y}+i\frac{\sin 2y}{\cosh 2x+\cos 2y}$$ ■

17. *Find the real and imaginary parts of the following*

(a) $\tan^{-1}(x+iy)$, (b) $\cot^{-1}(x+iy)$

➡ (a) Let $u+iv=\tan^{-1}(x+iy)$ ---(1)

$u-iv=\tan^{-1}(x-iy)$ ---(2)

Adding (1) and (2), we get

$$2u=\tan^{-1}(x+iy)+\tan^{-1}(x-iy)$$

Using $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ above equation becomes,

$$=\tan^{-1}\left(\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}\right)$$

$$=\tan^{-1}\left(\frac{2x}{1-(x^2-i^2y^2)}\right)=\tan^{-1}\left(\frac{2x}{1-(x^2+y^2)}\right)$$

$$2u=\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

$$u=\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

Subtracting (2) from (1), we get

$$2iv = \tan^{-1}(x+iy) - \tan^{-1}(x-iy)$$

Using $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ above equation becomes,

$$= \tan^{-1}\left(\frac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}\right) = \tan^{-1}\left(\frac{2iy}{1+x^2-i^2y}\right)$$

$$2iv = i\tan^{-1}\left(\frac{2y}{1+x^2+y^2}\right) \qquad (\because \tan^{-1}(ix) = i\tan^{-1}x)$$

$$2v = \tan^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

$$v = \frac{1}{2}\tan^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

(b) Let $u+iv = \cot^{-1}(x+iy)$ ---(1)

$u-iv = \cot^{-1}(x-iy)$ ---(2)

Adding equations (1) and (2), we get

$$2u = \cot^{-1}(x+iy) + \cot^{-1}(x-iy)$$

Using $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\left(\frac{xy-1}{x+y}\right)$ above equation becomes,

$$= \cot^{-1}\left(\frac{(x+iy)(x-iy)-1}{x+iy+x-iy}\right) = \cot^{-1}\left(\frac{x^2-i^2y^2-1}{2x}\right)$$

$$2u = \cot^{-1}\left(\frac{x^2+y^2-1}{2x}\right)$$

$$u = \frac{1}{2}\cot^{-1}\left(\frac{x^2+y^2-1}{2x}\right)$$

Subtracting (2) from (1), we get

$$2iv = \cot^{-1}(x+iy) - \cot^{-1}(x-iy)$$

Using $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\left(\frac{xy+1}{x-y}\right)$ above equation becomes,

$$= \cot^{-1}\left(\frac{(x+iy)(x-iy)+1}{x+iy-(x+iy)}\right) = \cot^{-1}\left(\frac{x^2-i^2y^2+1}{x+iy-x+iy}\right)$$

$$= \cot^{-1}\left(\frac{x^2+y^2+1}{2iy}\right) = \cot^{-1}\left(\frac{-i(x^2+y^2+1)}{2y}\right)$$

$$2iv = -i\cot^{-1}\left(\frac{x^2+y^2+1}{2y}\right) \qquad (\because \cot^{-1}(ix) = i\cot^{-1}x)$$

$$v = -\frac{1}{2}\cot^{-1}\left(\frac{x^2+y^2+1}{2y}\right)$$ ■

18. *Prove that* $\sinh^{-1}x = \log_e\left(x+\sqrt{x^2+1}\right)$

➡ Let $y = \sinh^{-1}x$

$$\Rightarrow x = \sinh y = \frac{1}{2}\left(e^y - e^{-y}\right) = \frac{1}{2}\left(e^y - \frac{1}{e^y}\right)$$

$$x = \frac{e^{2y}-1}{2e^y}$$

$$2xe^y = e^{2y}-1$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Since e^y is positive it follows that

$$e^y = x + \sqrt{x^2+1}$$

$$y = \log_e\left(x+\sqrt{x^2+1}\right)$$

Therefore,

$$\sinh^{-1}x = \log_e\left(x+\sqrt{x^2+1}\right)$$ ■

19. *Prove that* $\cosh^{-1} x = \log_e\left(x+\sqrt{x^2-1}\right)$

➡ Let $y = \cosh^{-1} x$

$$\Rightarrow x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right) = \frac{1}{2}\left(e^y + \frac{1}{e^y}\right)$$

$$x = \frac{e^{2y}+1}{2e^y}$$

$$2xe^y = e^{2y}+1$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$$\Rightarrow e^y = x + \sqrt{x^2-1}$$

$$y = \log_e\left(x+\sqrt{x^2-1}\right)$$

Therefore, $\cosh^{-1} x = \log_e\left(x+\sqrt{x^2-1}\right)$ ■

20. *Prove that* $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

➡ Let $y = \tanh^{-1} x$

$$\Rightarrow x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$= \frac{e^y - \frac{1}{e^y}}{e^y + \frac{1}{e^y}} = \frac{\left(e^{2y}-1\right)/e^y}{\left(e^{2y}+1\right)/e^y}$$

$$x = \frac{e^{2y}-1}{e^{2y}+1}$$

$$x\left(e^{2y}+1\right) = e^{2y}-1$$

$$e^{2y} - 1 - xe^{2y} - x = 0$$

$$(1-x)e^{2y} = 1+x$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$e^{y} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$y = \log_e\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$y = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

Therefore,

$$\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad ■$$

21. *Prove that* $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

➡ Let $z = \cos\theta + i\sin\theta$

$$\frac{1}{z} = \cos\theta - i\sin\theta$$

$$z^n = \cos n\theta + i\sin n\theta$$

$$\frac{1}{z^n} = \cos n\theta - i\sin n\theta \Rightarrow z + \frac{1}{z} = 2\cos\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

Therefore, $\cos^4\theta = \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^4$

$$= \frac{1}{2^4}\left(z + \frac{1}{z}\right)^4$$