DOUBLE INTEGRALS

Double integrals can be evaluated by expressing it in terms of two single integrals. If the region *R* is bounded by curves
$x = x_1, x = x_2$ and $y = y_1, y = y_2$ then

$$I = \iint_R f(x, y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\,dx\,dy$$

To evaluate this double integral, we proceed as follows:

Case (i) **Let** x_1, x_2 and y_1, y_2 be constants, then it does not matter, whether we first integrate with respect to *x* and, then with respect to *y* or vice versa.

Case (ii) Let x_1, x_2 be constants and y_1, y_2 be functions of *x*, where $y_1 = f_1(x)$ and $y_2 = f_2(x)$. In this case *f* (*x*, *y*) is first integrated with respect to *y*, treating *x* as constant between the limits $y_1 = f_1(x)$, $y_2 = f_2(x)$ and, then the resulting expression is integrated with respect to *x* between the limits x_1 and x_2.

$$\text{i.e. } I = \int_{x_1}^{x_2}\left[\int_{f_1(x)}^{f_2(x)} f(x, y)\,dy\right]dx$$

Case (iii) Let y_1, y_2 be constants and x_1, x_2 be functions of *y*, where $x_1 = g_1(y), x_2 = g_2(y)$. In this case *f* (*x*, *y*) is first integrated with respect to *x* treating *y* as constant between the limits $x_1 = g_1(y)$ and $x_2 = g_2(y)$ and, then the resulting expression is integrated with respect to *y* between the limits y_1 and y_2

$$\text{i.e., } I = \int_{y_1}^{y_2}\left[\int_{g_1(y)}^{g_2(y)} f(x, y)\,dx\right]dy$$

Case (iv) Let R be the region bounded by the simple closed curve C

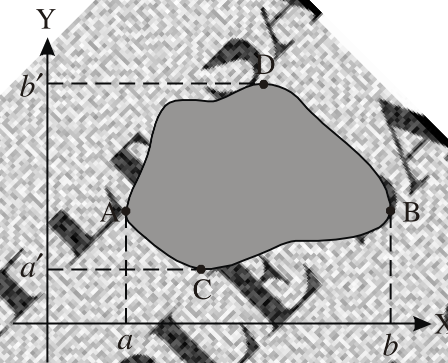

Figure 3.1

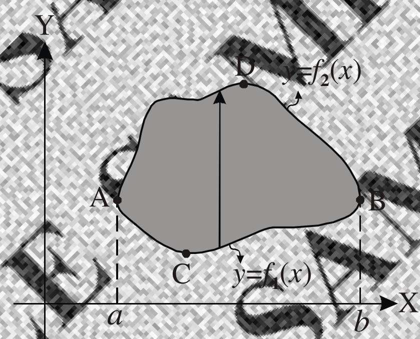

Figure 3.1(a)

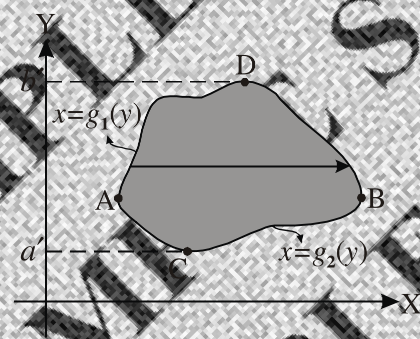

Figure 3.1(b)

Let a be the least and b be the greatest values of x in this region. Now, let $y = f_1(x)$ be the equation of the curve ACB and $y = f_2(x)$ be the equation of the curve ADB. Thus x varies from a to b and y varies from $f_1(x)$ to $f_2(x)$

Therefore, $I = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\,dy\,dx$

Using case (ii), we can evaluate the above integral.

We can construct the integral I in an alternative way by taking constant limits for y as explained below.

Let a' be the least and b' be the greatest values of y in the region. Now let $x = g_1(y)$ be the equation of the curve DAC and $x = g_2(y)$ be the equation of the curve DBC. Therefore, y varies from a' to b' and x varies from $g_1(y)$ to $g_2(y)$

$$\therefore I = \int_{a'}^{b'} \int_{g_1(y)}^{g_2(y)} f(x, y)\,dx\,dy$$

Using case (iii), we can evaluate the above integral.

Worked Examples

1. *Evaluate* $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$

➡ $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \int_0^1 \frac{dxdy}{\sqrt{1-x^2}.\sqrt{1-y^2}}$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\int_0^1 \frac{1}{\sqrt{1-x^2}} dx \right] dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x \right]_0^1 dy = \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\frac{\pi}{2} - 0 \right] dy$$

$$= \frac{\pi}{2} \sin^{1} y \Big]_0^1 = \frac{\pi}{2}\left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{4}$$ ■

2. *Evaluate* $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\,dy\,dx$

➡ Let $I = \int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\,dy\,dx$

$$= \int_0^{\pi/2} \left[-\cos(x+y) \right]_0^{\frac{\pi}{2}} dx \qquad \left[\int \sin(a+x)\,dx = -\cos(a+x) \right]$$

$$= -\int_0^{\pi/2} \left[\cos\left(x + \frac{\pi}{2} \right) - \cos x \right] dx = -\int_0^{\pi/2} \left[-\sin x - \cos x \right] dx$$

$$= \int_0^{\pi/2} (\sin x + \cos x)\, dx = -\cos x + \sin x \Big]_0^{\frac{\pi}{2}}$$

$$= -(0-1) + (1-0) = 2$$ ■

3. Evaluate $\int_0^1\int_{x^2}^{x} (x^2+3y+2)\,dydx$

➡ $$\int_0^1\int_{x^2}^{x} (x^2+3y+2)\,dydx = \int_0^1 \left.(x^2+2)y + \frac{3y^2}{2}\right|_{x^2}^{x} dx$$

$$= \int_0^1 (x^2+2)(x-x^2) + \frac{3}{2}(x^2-x^4)\,dx$$

$$= \int_0^1 \left[(x^3+2x-x^4-2x^2) + \frac{3}{2}x^2 - \frac{3}{2}x^4\right] dx$$

$$= \frac{1}{2}\int_0^1 (2x^3+4x-2x^4-4x^2+3x^2-3x^4)\,dx$$

$$= \frac{1}{2}\int_0^1 (-5x^4+2x^3-x^2+4x)\,dx$$

$$= \frac{1}{2}\left[\frac{-5x^5}{5}+\frac{2x^4}{4}-\frac{x^3}{3}+\frac{4x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left[-(1-0)+\frac{1}{2}(1-0)-\frac{1}{3}(1-0)+2(1-0)\right]$$

$$= \frac{1}{2}\left[-1+\frac{1}{2}-\frac{1}{3}+2\right]$$

$$= \frac{1}{2}\left[\frac{-6+3-2+12}{6}\right] = \frac{7}{12}$$ ■

4. *Evaluate* $\int_0^1\int_0^{1-x} (x+y)^2\, dydx$

➡ Let $I = \int_0^1\int_0^{1-x} (x+y)^2\, dydx$

$$= \int_0^1 \left[\frac{(x+y)^3}{3} \right]_0^{(1-x)} dx$$

$$= \frac{1}{3}\int_0^1 \left[(x+1-x)^3 - (x+0)^3 \right] dx = \frac{1}{3}\int_0^1 \left(1-x^3\right) dx$$

$$= \frac{1}{3}\left[x - \frac{x^4}{4} \right]_0^1 = \frac{1}{3}\left[(1-0) - \frac{1}{4}(1-0) \right] = \frac{1}{3}\left(1 - \frac{1}{4}\right)$$

$$= \frac{1}{4}$$ ■

5. *Evaluate* $\int_1^2 \int_0^{2-y} xy\,dx\,dy$

➡ Let $I = \int_1^2 \int_0^{2-y} xy\,dx\,dy$

$$= \int_1^2 \left[y\frac{x^2}{2} \right]_0^{2-y} dy = \frac{1}{2}\int_1^2 y\left[(2-y)^2 - 0\right] dy$$

$$= \frac{1}{2}\int_1^2 y\left[4 + y^2 - 4y\right] dy$$

$$= \frac{1}{2}\int_1^2 \left(4y + y^3 - 4y^2\right) dy = \frac{1}{2}\left[\frac{4y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3} \right]_1^2$$

$$= \frac{1}{2}\left[2\left(2^2 - 1^2\right) + \frac{1}{4}\left(2^4 - 1^4\right) - \frac{4}{3}\left(2^3 - 1^3\right) \right]$$

$$= \frac{5}{24}$$ ■

6. *Evaluate* $\int_0^1 \int_0^{\sqrt{1-x^2}} xye^{x^2}\,dy\,dx$

➡ Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} xye^{x^2}\,dy\,dx$

$$= \int_0^1 x e^{x^2} \frac{y^2}{2}\Bigg]_0^{\sqrt{1-x^2}} dx = \frac{1}{2}\int_0^1 x e^{x^2}\left(1-x^2-0\right)dx$$

$$= \frac{1}{2}\int_0^1 x\left(1-x^2\right)e^{x^2}dx$$

Put $x^2 = t \Rightarrow 2xdx\ dt \Rightarrow xdx = \frac{dt}{2}$

If $x = 0 \Rightarrow t = 0$, if $x = 1 \Rightarrow t = 1$

$$\therefore I = \frac{1}{2}\int_0^1 (1-t)\, e^t \frac{dt}{2}$$

$$= \frac{1}{4}\left\{\left[(1-t)\, e^t\right]_0^1 - \int_0^1 (-1)\, e^t\, dt\right\} \qquad \text{[By integration by parts]}$$

$$= \frac{1}{4}\left[(0-1) + e^t\right]_0^1 = \frac{1}{4}\left[-1 + (e-1)\right]$$

$$= \frac{1}{4}(e-2)$$

■

7. *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$

➡ Let $I = \int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$

$$= \int_0^1 x\frac{y^2}{2}\Bigg|_x^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 x\left(x - x^2\right)dx$$

$$= \frac{1}{2}\int_0^1 \left(x^2 - x^3\right)dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{2}\left[\frac{1}{12}\right]$$

$$= \frac{1}{24}$$

■

8. *Evaluate* $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xydydx$

➡ Let $I = \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xydydx$

$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{2ax-x^2}} dx = \frac{1}{2}\int_0^{2a} x\left[\left(2ax-x^2\right)-0\right]dx$$

$$= \frac{1}{2}\int_0^{2a}\left(2ax^2-x^3\right)dx = \frac{1}{2}\left[2a\frac{x^3}{3}-\frac{x^4}{4}\right]_0^{2a}$$

$$= \frac{1}{2}\left[\frac{2a}{3}\left[(2a)^3-0\right]-\frac{1}{4}\left((2a)^4-0\right)\right]$$

$$= \frac{1}{2}\left[\frac{16a^4}{3}-\frac{16a^4}{4}\right] = \frac{16a^4}{2}\left[\frac{1}{3}-\frac{1}{4}\right]$$

$$= 8a^4\left(\frac{4-3}{12}\right) = \frac{2a^4}{3}$$ ■

9. *Evaluate* $\iint_R \left(x^2+y^2\right)dxdy$ *when R is the triangle bounded by the lines* $y=0$, $y=x$ *and* $x=1$

➡

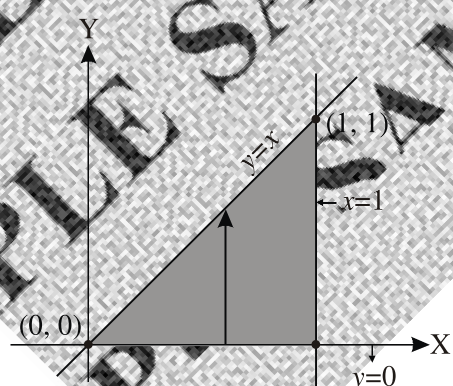

Figure 3.2

Here the minimum value of x is *0* and maximum value is *1*. Therefore x varies from *0* to *1*, and y varies from the line $y = 0$ to the line $y = x$

$$\therefore x : 0 \to 1,\ y : 0 \to x$$

$$\therefore \iint_A \left(x^2 + y^2\right) dxdy = \int_{x=0}^{1} \int_{y=0}^{x} \left(x^2 + y^2\right) dydx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^x dx$$

$$= \int_0^1 \left[x^2 (x - 0) + \frac{1}{3}\left(x^3 - 0\right) \right] dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{3} \right) dx$$

$$= \frac{4}{3} \int_0^1 x^3 dx = \frac{4}{3} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{3}(1 - 0)$$

$$= \frac{1}{3}$$

10. *Evaluate* $\iint_A xydxdy$ *when A is area bounded by the circle* $x^2 + y^2 = a^2$ *in the first quadrant.*

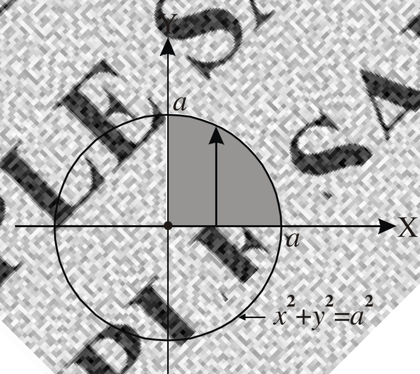

Figure 3.3

Here the minimum value of x is 0 and the maximum value is a. Therefore, $x:0 \to a$ and y varies from the line $y = 0$ to the circle $x^2 + y^2 = a^2$

$$\therefore y^2 = a^2 - x^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$$

We take $y = \sqrt{a^2 - x^2}$ because y is positive above the x-axis

$$\therefore y : 0 \to \sqrt{a^2 - x^2}$$

$$\therefore \iint_A xydxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xydydx$$

$$= \int_0^a x\frac{y^2}{2}\bigg]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x\left[a^2 - x^2 - 0\right]dx$$

$$= \frac{1}{2}\int_0^a \left(a^2x - x^3\right)dx = \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{8}$$

11. *Evaluate* $\iint_R xy\, dxdy$ *where R is the region bounded by x-axis, the ordinate x=2a and the parabola* $x^2 = 4ay$

➡

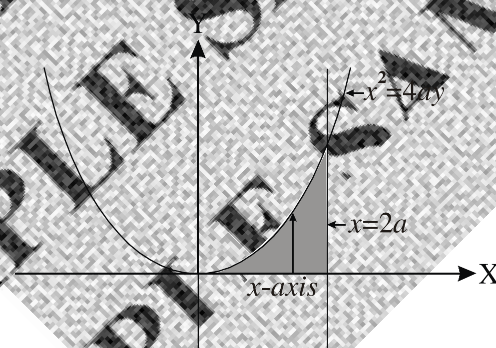

Figure 3.4

Here the minimum value of x is *0* and maximum value is $2a \Rightarrow x : 0 \rightarrow 2a$ and y varies from x axis to the parabola $x^2 = 4ay \Rightarrow y : 0 \rightarrow \frac{x^2}{4a}$

$$\iint_R xydxdy = \int_0^{2a} \int_0^{x^2/4a} xydydx$$

$$= \int_0^{2a} x\frac{y^2}{2}\bigg|_0^{x^2/4a} dx$$

$$= \frac{1}{2}\int_0^{2a} \frac{x^5}{16a^2} dx$$

$$= \frac{1}{32a^2}\frac{x^6}{6}\bigg|_0^{2a} dx$$

$$= \frac{1}{32a^2 \times 6}(2a)^6 = \frac{a^4}{3}$$

12. *If R is the region bounded by the parabolas* $x = y^2$ *and* $y = x^2$ *show that* $\iint_R xy(x+y)dxdy = \frac{3}{28}$

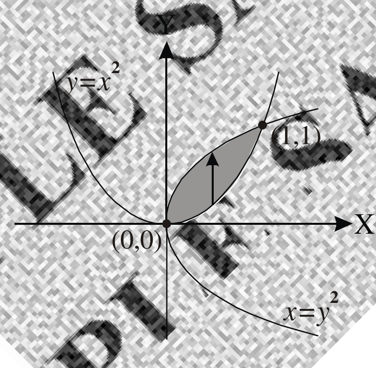

Figure 3.5

The given curves are

$y = x^2$ ---(1)

$x = y^2$ ---(2)

To find the points of intersection we substitute (2) in (1)

$\therefore y = \left(y^2\right)^2 = y^4$

$y - y^4 = 0$

$y\left(1 - y^3\right) = 0$

$\therefore y = 0$ or y = 1

$\therefore$ If $y = 0 \Rightarrow x = 0$ if $y = 1 \Rightarrow x = 1$

Therefore, the parabolas interacts at (0,0) and (1,1) $\therefore x : 0 \to 1$ and y varies from the parabola $y = x^2$ to the parabola $y^2 = x$

i.e., $y : x^2 \to \sqrt{x}$

$$\iint_R xy(x+y)\,dxdy = \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x+y)\,dydx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left(x^2y + xy^2\right)dydx = \int_0^1 x^2\frac{y^2}{2} + x\frac{y^3}{3}\Bigg|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\frac{x^2}{2}\left(x - x^4\right) + \frac{x}{3}\left(\left(\sqrt{x}\right)^3 - x^6\right)\right]dx$$

$$= \int_0^1 \left(\frac{x^3}{2} - \frac{x^6}{2} + \frac{x^{5/2}}{3} - \frac{x^7}{3}\right)dx$$

$$= \frac{1}{2}\cdot\frac{x^4}{4} - \frac{1}{2}\cdot\frac{x^7}{7} + \frac{1}{3}\cdot\frac{x^{7/2}}{(7/2)} - \frac{1}{3}\cdot\frac{x^8}{8}\Bigg]_0^1$$

$$= \frac{1}{2\cdot 4} - \frac{1}{2\cdot 7} + \frac{2}{3\cdot 7} - \frac{1}{3\cdot 8}$$

$$= \frac{7\cdot 3 - 3\cdot 4 + 16 - 7}{7\cdot 3\cdot 8} = \frac{21 - 12 + 9}{7\cdot 3\cdot 8} = \frac{18}{7\cdot 3\cdot 8}$$

$$= \frac{3}{28}$$ ■

13. *Evaluate* $\iint\limits_R y^2\,dxdy$ *where R is the region bounded by the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

➡

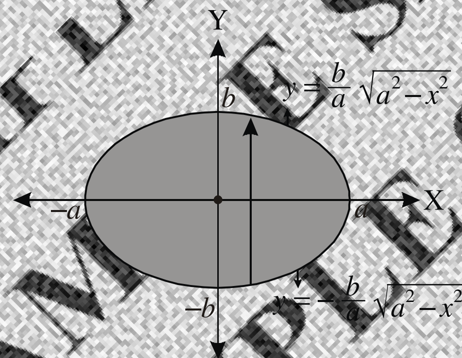

Figure 3.6

Here x varies from $-a$ to a and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$y:-\frac{b}{a}\sqrt{a^2-x^2}\Rightarrow\frac{b}{a}\sqrt{a^2-x^2}$$

Therefore,

$$\iint\limits_R y^2\,dxdy=\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2\,dydx$$

$$= \int_{-a}^{a} \frac{y^3}{3}\Bigg|_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{3}\int_{-a}^{a}\left[\frac{b^3}{a^3}\left(a^2-x^2\right)^{\frac{3}{2}} + \frac{b^3}{a^3}\left(a^2-x^2\right)^{\frac{3}{2}}\right]dx$$

$$= \frac{1}{3}\int_{-a}^{a}\frac{2b^3}{a^3}\left(a^2-x^2\right)^{3/2}dx = \frac{2b^3}{3a^3}\int_{-a}^{a}\left(a^2-x^2\right)^{3/2}dx$$

$$= \frac{2b^3}{3a^3}2\int_{0}^{a}\left(a^2-x^2\right)^{3/2}dx = \frac{4b^3}{3a^3}\int_{0}^{a}\left(a^2-x^2\right)^{3/2}dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

If $x = 0 \Rightarrow \theta = 0$, if $x = a \Rightarrow \theta = \pi/2$

$$= \frac{4b^3}{3a^3}\int_{0}^{\pi/2}\left(a^2 - a^2\sin^2\theta\right)^{\frac{3}{2}} a\cos\theta\, d\theta$$

$$= \frac{4b^3}{3a^3}\int_{0}^{\pi/2} a^3\cos^3\theta \cdot a\cos\theta\, d\theta$$

$$= \frac{4b^3}{3}a\int_{0}^{\pi/2}\cos^4\theta\, d\theta = \frac{4b^3a}{3}\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{ab^3\pi}{4}$$ ■

14. *Evaluate* $\iint_R dx\,dy$ *where* R *is the region bounded by the lines* $y = x,\ x + y = 4,\ y = 1$ *and* $y = 0$

➡

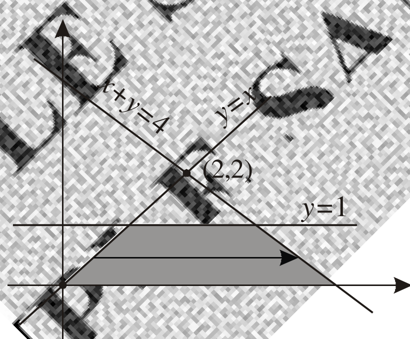

Figure 3.7

Here y varies from 0 to 1 and x varies from the line $y = x$ to the line $x + y = 4$

i.e., $y : 0 \to 1,\ x : y \to 4 - y$

Therefore, $\iint_R dx\,dy = \int_0^1 \int_y^{4-y} dx\,dy$

$$= \int_0^1 x \Big|_y^{4-y} dy = \int_0^1 (4 - y - y)\,dy$$

$$= \int_0^1 (4 - 2y)\,dy = 4y - \frac{2y^2}{2}\Big|_0^1$$

$$= 4(1-0) - (1-0) = 4 - 1 = 3$$

15. *Evaluate* $\iint_R dxdy$ *where R is the region bounded between the parabola* $y = x^2$ *and the line* $x + y = 2$

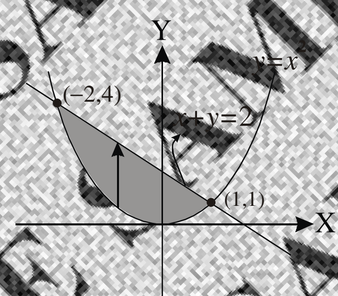

Figure 3.8

$y = x^2$ ---(1)

$x + y = 2$ ---(2)

Substituting (1) in (2), we get

$$x + x^2 = 2$$