(Homogeneous function means power of the variables in each term should be same)

(ii) In *the differential equation* $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ *if the power of the variables of* $f(x,y)$ *is equal to the power of the variables of* $g(x,y)$, *then the equation is called Homogeneous equation.*

Worked Examples

1. *Solve* $\frac{dy}{dx}=1+\frac{y}{x}+\left(\frac{y}{x}\right)^2$

➡ $\frac{dy}{dx}=1+\frac{y}{x}+\left(\frac{y}{x}\right)^2$ ---(1)

(The RHS of the above equation is a function of y/x, *hence it is a homogeneous differential equation.)*

Put $\frac{y}{x}=v$, i.e., $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

∴ (1) becomes, $v+x\frac{dv}{dx}=1+v+v^2$

$$x\frac{dv}{dx}=1+v^2$$

$$\frac{dv}{1+v^2}=\frac{dx}{x}$$

$$\int\frac{dv}{1+v^2}=\int\frac{dx}{x}+c$$

$$\tan^{-1}v=\log x+c$$

$$\tan^{-1}(y/x)=\log x+c$$ ■

2. *Solve* $\frac{dy}{dx}=\frac{y}{x}+\sin\frac{y}{x}$

➡ $\frac{dy}{dx}=\frac{y}{x}+\sin\frac{y}{x}$ ---(1)

Put $\frac{y}{x} = v$ i.e., $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $\quad v + x\frac{dv}{dx} = v + \sin v$

$$x\frac{dv}{dx} = \sin v$$

$$\frac{dv}{\sin v} = \frac{dx}{x}$$

$$\int \frac{dv}{\sin v} = \int \frac{dx}{x} + c$$

$$\int \operatorname{cosec} v\, dv = \log x + c$$

$$\log(\operatorname{cosec} v - \cot v) = \log x + c$$

$$\log(\operatorname{cosec}(y/x) - \cot(y/x)) = \log x + c \quad ■$$

3. *Solve* $x^2 y dx - (x^3 + y^3) dy = 0$

➡ $x^2 y dx - (x^3 + y^3) dy = 0$

[*This is in the form of f (x, y) dx + g(x, y) dy=0.*
Here f (x, y) and g (x, y) are homogenous functions of degree three hence, it is homogenous equation]

$$x^2 y dx = (x^3 + y^3) dy$$

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \quad \text{---(1)}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $\quad v + x\frac{dv}{dx} = \frac{x^2(vx)}{x^3 + (vx)^3} = \frac{x^3 v}{x^3(1 + v^3)}$

$$v + x\frac{dv}{dx} = \frac{v}{1 + v^3}$$

$$\therefore x\frac{dv}{dx} = \frac{v}{1 + v^3} - v = \frac{v - v - v^4}{1 + v^3}$$

$$\therefore x\frac{dv}{dx} = -\frac{v^4}{1+v^3}$$

$$\frac{1+v^3}{v^4}dv = -\frac{dx}{x}$$

$$\int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\int\frac{dx}{x}+c$$

$$\int\left(v^{-4}+\frac{1}{v}\right)dv = -\log x + c$$

$$\frac{v^{-3}}{-3}+\log v+\log x = c$$

$$-\frac{1}{3v^3}+\log v+\log x = c$$

$$-\frac{1}{3v^3}+\log(vx) = c$$

$$-\frac{1}{3(y/x)^3}+\log y = c$$

$$\therefore \log y - \frac{x^3}{3y^3} = c$$

4. *Solve* $x\frac{dy}{dx} = y(\log y - \log x + 1)$

➡ $x\frac{dy}{dx} = y(\log y - \log x + 1)$

$$\frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right)+1\right) \quad \text{---(1)}$$

Put $\frac{y}{x} = v$ i.e., $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = v(\log v + 1)$

$$v + x\frac{dv}{dx} = v\log v + v$$

$$x\frac{dv}{dx} = v\log v$$

$$\frac{dv}{v\log v} = \frac{dx}{x}$$

$$\int \frac{dv}{v\log v} = \int \frac{dx}{x} + c$$

Put $\log v = t \Rightarrow \frac{1}{v}dv = dt$

$$\int \frac{dt}{t} = \log x + c$$

$$\log t = \log x + \log c_1$$

$$\log t = \log(c_1 x)$$

$$t = c_1 x$$

$$\log v = c_1 x$$

$$\log\left(\frac{y}{x}\right) = c_1 x$$

5. *Solve* $(x - y\log y + y\log x)dx + x(\log y - \log x)dy = 0$

➡ $(x - y\log y + y\log x)dx + x(\log y - \log x)dy = 0$

$$x - y\log y + y\log x dx = -x(\log y - \log x)dy$$

$$[x - y(\log y - \log x)]\ dx = -x(\log y - \log x)dy$$

$$\left(x - y\log\left(\frac{y}{x}\right)\right)dx = -x\log\left(\frac{y}{x}\right)dy$$

$$\left[1 - \frac{y}{x}\log\left(\frac{y}{x}\right)\right] = -\log\left(\frac{y}{x}\right)\frac{dy}{dx}$$

Put $\frac{y}{x} = v$ i.e., $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$(1 - v\log v) = -\log v\left[v + x\frac{dv}{dx}\right]$$

$$1 - v\log v = -v\log v - x\log v\frac{dv}{dx}$$

$$1 = -x\log v\frac{dv}{dx}$$

$$\frac{dx}{x}+\log v\,dv=0$$

$$\int\frac{dx}{x}+\int\log v\,dv=c$$

$$\log x+v(\log v-1)=c$$

$$\log x+\frac{y}{x}\left[\log\left(\frac{y}{x}\right)-1\right]=c$$

■

6. *Solve* $\dfrac{x}{y}\dfrac{dy}{dx}=\dfrac{x\cos(y/x)+y\sin(y/x)}{y\sin(y/x)-x\cos(y/x)}$

➡ $\dfrac{x}{y}\dfrac{dy}{dx}=\dfrac{x\cos(y/x)+y\sin(y/x)}{y\sin(y/x)-x\cos(y/x)}$

$$\frac{dy}{dx}=\frac{xy\cos(y/x)+y^2\sin(y/x)}{xy\sin(y/x)-x^2\cos(y/x)} \quad \text{---(1)}$$

Put $y=vx \Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\therefore$ (1) becomes, $v+x\dfrac{dv}{dx}=\dfrac{x(vx)\cos\left(\frac{vx}{x}\right)+(vx)^2\sin\left(\frac{vx}{x}\right)}{x(vx)\sin\left(\frac{vx}{x}\right)-x^2\cos\left(\frac{vx}{x}\right)}$

$$=\frac{x^2\left[v\cos v+v^2\sin v\right]}{x^2\left[v\sin v-\cos v\right]}$$

$$v+x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}$$

$$x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}-v$$

$$=\frac{v\cos v+v^2\sin v-v^2\sin v+v\cos v}{v\sin v-\cos v}$$

$$x\frac{dv}{dx}=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

$$\int\left(\frac{v\sin v}{v\cos v} - \frac{\cos v}{v\cos v}\right)dv = 2\int\frac{dx}{x} + c$$

$$\int\left(\tan v - \frac{1}{v}\right)dv = 2\log x + c$$

$$\log(\sec v) - \log v = \log x^2 + \log c_1 \qquad [c = \log c_1]$$

$$\log\left[\frac{\sec v}{v}\right] = \log\left(c_1 x^2\right)$$

$$\frac{\sec v}{v} = c_1 x^2$$

$$\sec v = c_1 x^2 v$$

$$\sec\left(\frac{y}{x}\right) = c_1 x^2\left(\frac{y}{x}\right)$$

$$\therefore \sec\left(\frac{y}{x}\right) = c_1 xy$$

7. *Solve* $[x\sin(y/x) - y\cos(y/x)]dx + x\cos(y/x)dy = 0$

➡ $[x\sin(y/x) - y\cos(y/x)]dx + x\cos(y/x)dy = 0$

$$x\cos(y/x)dy = -[x\sin(y/x) - y\cos(y/x)]dx$$

$$\frac{dy}{dx} = -\frac{[x\sin(y/x) - y\cos(y/x)]}{x\cos(y/x)}$$

$$\frac{dy}{dx} = -\tan\left(\frac{y}{x}\right) + \frac{y}{x} \qquad \text{---(1)}$$

Put $\frac{y}{x} = v$ i.e., $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = -\tan v + v$

$$x\frac{dv}{dx} = -\tan v$$

$$\frac{dv}{\tan v} = -\frac{dx}{x}$$

$$\int \cot v\, dv = -\int \frac{dx}{x} + c$$

$$\log(\sin v) = -\log x + c$$

$$\log(\sin v) + \log x = \log c_1, \text{ where } \log c_1 = c$$

$$\log(x \sin v) = \log c_1$$

$$x \sin v = c_1$$

$$x \sin\left(\frac{y}{x}\right) = c_1$$

8. *Solve* $\left[x\tan(y/x) - y\sec^2(y/x)\right] dx + x\sec^2(y/x)dy = 0$

➡ $\left[x\tan(y/x) - y\sec^2(y/x)\right] dx + x\sec^2(y/x)dy = 0$

$$x\sec^2(y/x)dy = -\left(x\tan(y/x) - y\sec^2(y/x)\right)dx$$

$$\frac{dy}{dx} = \frac{y\sec^2(y/x) - x\tan(y/x)}{x\sec^2(y/x)} = \frac{y}{x} - \frac{\tan(y/x)}{\sec^2(y/x)} \quad \text{---(1)}$$

Put $\frac{y}{x} = v$ i.e., $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = v - \frac{\tan v}{\sec^2 v}$

$$x\frac{dv}{dx} = -\frac{\tan v}{\sec^2 v}$$

$$\frac{\sec^2 v}{\tan v} dv = -\frac{dx}{x}$$

$$\int \frac{\sec^2 v}{\tan v} dv = -\int \frac{dx}{x} + c$$

Put $\tan v = t \Rightarrow \sec^2 v\, dv = dt$

$$\therefore \int \frac{dt}{t} = -\log x + c$$

$$\log t = -\log x + c$$

$$\log t + \log x = c$$

$$\log(xt) = \log c_1$$
$$xt = c_1$$
$$x\tan v = c_1$$
$$x\tan\left(\frac{y}{x}\right) = c_1$$

■

9. *Solve* $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

➡ $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$$y^2 = xy\frac{dy}{dx} - x^2\frac{dy}{dx}$$
$$y^2 = \left(xy - x^2\right)\frac{dy}{dx}$$
$$\therefore \frac{dy}{dx} = \frac{y^2}{xy - x^2} \qquad \text{---(1)}$$

Which is a homogeneous equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

(1) becomes, $v + x\frac{dv}{dx} = \frac{(vx)^2}{x(vx) - x^2} = \frac{v^2x^2}{x^2(v-1)}$

$$v + x\frac{dv}{dx} = \frac{v^2}{v-1}$$
$$x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1}$$
$$x\frac{dv}{dx} = \frac{v}{v-1}$$
$$\frac{v-1}{v}dv = \frac{dx}{x}$$
$$\int\frac{v-1}{v}dv = \int\frac{dx}{x} + c$$

$$\int\left(1-\frac{1}{v}\right)dv = \log x + c$$

$$\therefore v - \log v = \log x + c$$

$$v = \log v + \log x + c$$

$$v = \log(vx) + c$$

$$y/x = \log y + c$$

■

10. *Solve* $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$

➡ $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$

$$y - x = y\frac{dy}{dx} + x\frac{dy}{dx}$$

$$y - x = (x + y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y - x}{x + y} \qquad \text{---(1)}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

(1) becomes $v + x\frac{dv}{dx} = \frac{vx - x}{x + vx} = \frac{(v-1)x}{(1+v)x}$

$$v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$x\frac{dv}{dx} = \frac{v-1}{v+1} - v$$

$$= \frac{v - 1 - v(v+1)}{v+1} = \frac{v - 1 - v^2 - v}{v+1}$$

$$= \frac{-1 - v^2}{1 + v}$$

$$x\frac{dv}{dx} = -\frac{1 + v^2}{1 + v}$$