11. *If* $y = x^n \log x$, *show that* $y_{n+1} = \dfrac{n!}{x}$

➡ $y = x^n \log x$

$$y_1 = x^n \frac{1}{x} + nx^{n-1} \log x$$

$$y_1 = x^{n-1} + nx^{n-1} \log x$$

Multiplying x on both sides, we get

$$xy_1 = x^n + nx^n \log x$$

$$xy_1 = x^n + ny$$

Differentiating n times, we get

$$xy_{n+1} + {}^nC_1 1 y_n = n! + ny_n$$

$$xy_{n+1} + n\ y_n = n! + ny_n = n!$$

$$y_{n+1} = \frac{n!}{x}$$ ■

12. *If* $y = x^{n-1} \log x$, *show that* $y_n = \dfrac{(n-1)!}{x}$

➡ $y = x^{n-1} \log x$

$$y_1 = x^{n-1} \frac{1}{x} + (n-1)x^{n-2} \log x$$

$$y_1 = x^{n-2} + (n-1)x^{n-2} \log x$$

Multiplying x on both sides, we get

$$xy_1 = x^{n-1} + (n-1)x^{n-1} \log x$$

$$xy_1 = x^{n-1} + (n-1)y$$

Differentiating $(n-1)$ times, we get

$$D^{n-1}(xy_1) = D^{n-1}\left[x^{n-1} + (n-1)y\right]$$

$$xD^{n-1}(y_1) + {}^{n-1}C_1(1)D^{n-2}(y_1) = (n-1)! + (n-1)y_{n-1}$$

$$xy_n + (n-1)y_{n-1} = (n-1)! + (n-1)y_{n-1}$$

$$xy_n = (n-1)!$$

$$y_n = \frac{(n-1)!}{x}$$ ■

13. *If* $y = \log\left[x + \sqrt{1+x^2}\right]$, *show that*

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$$

➡ $y = \log\left[x + \sqrt{1+x^2}\right]$

$$y_1 = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{1}{2\sqrt{1+x^2}} \times 2x\right]$$

$$y_1 = \frac{1}{x+\sqrt{1+x^2}} \frac{\left[\sqrt{1+x^2}+x\right]}{\sqrt{1+x^2}}$$

$\sqrt{1+x^2}\, y_1 = 1$

Squaring, on both sides, we get

$$(1+x^2)y_1^2 = 1$$

Differentiating again with respect to x, we get

$$(1+x^2)2y_1y_2 + 2xy_1^2 = 0$$

Dividing $2y_1$ on both sides, we get

$$(1+x^2)y_2 + xy_1 = 0$$

Differentiating n times, we get

$$(1+x^2)y_{n+2} + 2x\,{}^nC_1 y_{n+1} + 2\,{}^nC_2 y_n + xy_{n+1} + {}^nC_1 y_n = 0$$

$$(1+x^2)y_{n+2} + 2xn\,y_{n+1} + 2\frac{n(n-1)}{2!}y_n + xy_{n+1} + ny_n = 0$$

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n)y_n = 0$$

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$$ ■

14. *If* $y = \frac{\log x}{x}$, *prove that*

$$y_n = \frac{(-1)^n n!}{x^{n+1}}\left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots\ldots\ldots - \frac{1}{n}\right\}$$

➡ $y = \log x . \frac{1}{x}$

We know that $D^n(uv) = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \ldots\ldots\ldots + u_n v$

Put $u = \log x$

Therefore, $u_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

and $v = \dfrac{1}{x}$, therefore, $v_n = \dfrac{(-1)^n n!}{x^{n+1}}$

$$\therefore D^n\left(\log x.\frac{1}{x}\right) = \log x.\frac{(-1)^n n!}{x^{n+1}} + {}^nC_1\frac{1}{x}.\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$+{}^nC_2\left(\frac{-1}{x^2}\right)\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + \ldots\ldots\ldots + \frac{(-1)^{n-1}(n-1)!}{x^n}\frac{1}{x}$$

$$D^n\left(\frac{\log x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\log x + \frac{(-1)^{n-1}n(n-1)!}{x^{n+1}}$$

$$+(-1)^{n-1}\frac{n(n-1)}{2!}\frac{(n-2)!}{x^{n+1}} + \ldots\ldots\ldots + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\log x + \frac{(-1)^{n-1}n!}{x^{n+1}} + \frac{(-1)^{n-1}n!}{2x^{n+1}}$$

$$+\ldots\ldots\ldots + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x + \frac{(-1)^{-1}}{1} + \frac{(-1)^{-1}}{2} + \ldots\ldots\ldots + \frac{(-1)^{-1}(n-1)!}{n!}\right]$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - \frac{1}{1} - \frac{1}{2} - \ldots\ldots\ldots - \frac{(n-1)!}{n(n-1)!}\right]$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots\ldots\ldots - \frac{1}{n}\right]$$ ■

Exercises

1. *Find the n^{th} derivatives of the following functions*

a) x^2e^{ax} b) $e^x(ax+b)^3$

c) $x^3\cos x$ d) $x^3\log x$

e) $e^x\log x$ f) $x^2\sin x$

g) x^ne^x h) $x^2\log 3x$

2. *If* $y = x^2 e^x$, *prove that*
$$y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y.$$

3. *If* $x = \tan(\log y)$, *prove that*
$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0.$

4. *If* $y = (\sin^{-1} x)^2$, *prove that* $(1-x^2)y_2 - xy_1 - 2 = 0$, *differentiate the above equation n times with respect to x.*

5. *If* $y = (x^2-1)^n$, *prove that*
$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$

6. *If* $y = \sin(m \sin^{-1} x)$, *prove that*
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0.$

7. *If* $x = \sin t$ *and* $y = \sin pt$, *prove that* $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - p^2)y_n = 0.$

8. *If* $y = e^{a\cos^{-1} x}$, *prove that* $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0.$

9. *If* $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, *prove that*
$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0.$

10. *If* $y = (\sin^{-1} x)^2$, *prove that*
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$
Hence show that $y_n(0) = 0$ *if n is even,*
and $y_n(0) = (n-2)^2(n-4)^2 \ldots\ldots\ldots\ldots 5^2 \cdot 3^2 \cdot 1^2$ *if n is odd.*

11. *If* $y = \tan x$, *prove that*
$$y_n(0) - {}^n c_2 y_{n-2}(0) + {}^n c_4 y_{n-4}(0) - \ldots\ldots = \sin\left(\frac{n\pi}{2}\right).$$

12. *If* $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, *prove that*
$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0.$

13. *If* $y = a\left(x+\sqrt{x^2-1}\right)+b\left(x-\sqrt{x^2-1}\right)$ *prove that*
$\left(x^2-1\right)y_{n+2}+(2n+1)xy_{n+1}=0$

14. *If* $y=e^{x^2/2}\cos x$, *prove that*
$y_{2n+2}(0)-4ny_{2n}(0)+2n(2n-1)y_{2n-2}(0)=0$.

15. *If* $y=ax^{n+1}+\dfrac{b}{x^n}$, *prove that* $x^2y_{n+2}+2nxy_{n+1}-2ny_n=0$.

16. *If* $y=\left[\log\left(x+\sqrt{x^2+a^2}\right)\right]^2$, *prove that*
$\left(a^2+x^2\right)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$

17. *If* $y=e^{-x^2}$ *show that* $y_{n+1}+2xy_n+2ny_{n-1}=0$

18. *If* $y=a\cosh\left(\log x^m\right)+b\sinh\left(\log x^m\right)$ *prove that*
$x^2y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-m^2\right)y_n=0$

19. *If* $y=\cos\log\left(x^2-2x+1\right)$, *prove that*
$(x-1)^2y_{n+2}+(2n+1)(x-1)y_{n+1}+\left(n^2+4\right)y_n=0$

Answers

1. a) $a^{n-2}e^{ax}\left\{a^2x^2+2nax+n(n-1)\right\}$

b) $e^x\left\{(ax+b)^3+3an(ax+b)^2+3x(n-1)a^2(ax+b)+n(n-1)(n-2)a^3\right\}$

c) $x\left\{n^2-3n(n-1)\right\}\cos(x+n(\pi/2))+n\left\{3x^2-n(n-1)(n-2)\right\}\sin(x+n(\pi/2))$

d) $e^x\left\{\log x+{}^nC_1x^{-1}-{}^nC_2x^{-2}+{}^nC_3\,2!\,x^{-3}+\ldots+(-1)^{n-1}(n-1)!\,x^{-n}\right\}$

e) $\left(x^2-n^2+n\right)\sin(x+n(\pi/2))-2nx\cos(x+n(\pi/2))$

f) $e^x\left\{x^n+\dfrac{n^2}{1!}x^{n-1}+\dfrac{n^2(n-1)^2}{2!}x^{n-2}+\ldots+\dfrac{n^2(n-1)^2\ldots\ldots\ldots1^2}{n!}\right\}$

g) $\dfrac{2}{x^{n-2}}(-1)^{n-1}(n-3)!$

h) $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$

POLAR CURVES

Let *OX, OY* be the rectangular axes with origin *O*. Let *P* be any point specified by its Cartesian co-ordinates (x, y). The point P can also be specified in the form of (r,θ), where $OP = r$ and θ is the angle $X\hat{O}P$ measured in the anticlock wise direction. Here (r,θ) are called the Polar coordinates of the point *P*. In particular r is called the radius vector and θ is called the Polar angle.

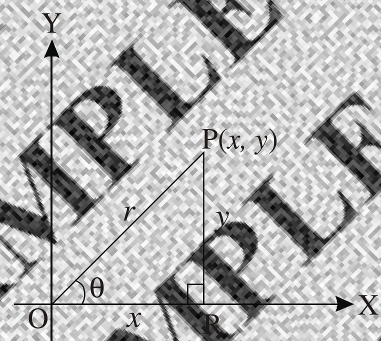

Figure 2.1

Now we can find the relations between Cartesian and Polar co-ordinates. From the right-angled triangle *OPR*, we have

$$\cos\theta = \frac{OR}{OP} = \frac{x}{r} \text{ and } \sin\theta = \frac{PR}{OP} = \frac{y}{r}$$

$$\therefore x = r\cos\theta \qquad \text{---(1)}$$

and $y = r\sin\theta$ ---(2)

$$\therefore x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$\therefore x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2} \qquad \text{---(3)}$$

and $\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta$

$$\therefore \theta = \tan^{-1}\left(y/x\right) \qquad \text{---(4)}$$

From the relations (1) and (2) we can find Cartesian co-ordinates when the Polar co-ordinates are given and from the relations (3) and (4) we can find the Polar co-ordinates when the Cartesian co-ordinates are given.

The point $P(r,\theta)$ varies along the curve when both r and θ vary. Hence the equation of the curve in Polar form is $r = f(\theta)$ or $f(r,\theta) = a$ where a is constant. The curve whose equation is given in the Polar form is called the Polar curve.

Angle between Radius Vector and the Tangent to the Polar Curve

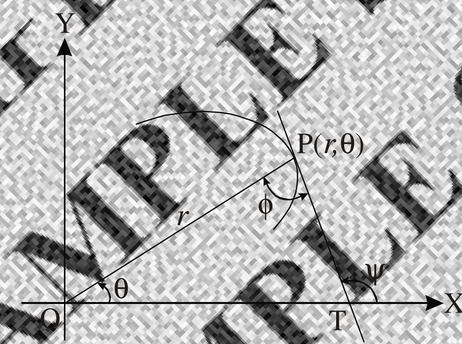

Figure 2.2

Let $r = f(\theta)$ be any Polar curve. Let P be any point on the curve, PT be the tangent at P to the curve, ψ be the angle between the tangent and the x-axis and ϕ be the angle between radius vector and tangent.

Therefore, slope of $PT = \tan\psi = \tan(\phi+\theta) = \dfrac{\tan\phi + \tan\theta}{1-\tan\phi\tan\theta}$ ---(1)

We have, $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{dx}{d\theta} = r(-\sin\theta) + \frac{dr}{d\theta}\cos\theta = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + (dr/d\theta)\sin\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta(dr/d\theta)\,[r(d\theta/dr) + \tan\theta]}{\cos\theta(dr/d\theta)\,[1 - r(d\theta/dr)\tan\theta]}$$

$$\frac{dy}{dx}=\frac{r(d\theta/dr)+\tan\theta}{1-r(d\theta/dr)\tan\theta} \qquad \text{---(2)}$$

From equations (1) and (2), we have

$$\frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta}=\frac{r(d\theta/dr)+\tan\theta}{1-r(d\theta/dr)\tan\theta}$$

$$\Rightarrow \tan\phi = r\frac{d\theta}{dr}$$

Worked Examples

1. *Find the angle between the radius vector and the tangent to the curve* $r=a(1-\cos\theta)$ *at the point* $\theta=\frac{\pi}{3}$

➡ $r=a(1-\cos\theta)$

$$\frac{dr}{d\theta}=a\sin\theta$$

$$\frac{d\theta}{dr}=\frac{1}{a\sin\theta}$$

$$r\frac{d\theta}{dr}=\frac{r}{a\sin\theta}=\frac{a(1-\cos\theta)}{a\sin\theta}$$

$$=\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}=\tan\left(\frac{\theta}{2}\right)$$

$$\tan\phi=r(d\theta/dr)=\tan(\theta/2)\Rightarrow\phi=\theta/2$$

At $\theta=\frac{\pi}{3},\quad \phi=\frac{\pi/3}{2}=\frac{\pi}{6}$ ■

2. *Find the angle between the radius vector and the tangent to the curve* $r^2\cos 2\theta=a^2$

➡ $r^2\cos 2\theta=a^2$

$$r^2(-\sin 2\theta)2+\cos 2\theta\, 2r\frac{dr}{d\theta}=0$$

$$2r\cos 2\theta\frac{dr}{d\theta}=2r^2\sin 2\theta$$

$$\frac{dr}{d\theta} = \frac{r\sin 2\theta}{\cos 2\theta} = \frac{\cos 2\theta}{r\sin 2\theta}$$

$$r\frac{d\theta}{dr} = \cot 2\theta$$

Therefore, $\tan\phi = r(d\theta/dr) = \cot 2\theta = \tan((\pi/2) - 2\theta)$

$$\Rightarrow \phi = (\pi/2) - 2\theta$$ ■

3. *Find the angle between the radius vector and the tangent to the curve* $r^m = a^m(\cos m\theta + \sin m\theta)$

➨ $r^m = a^m(\cos m\theta + \sin m\theta)$

$$mr^{m-1}\frac{dr}{d\theta} = a^m(-\sin m\theta \cdot m + \cos m\theta \cdot m)$$

$$mr^{m-1}\frac{dr}{d\theta} = ma^m(\cos m\theta - \sin m\theta)$$

$$\frac{dr}{d\theta} = \frac{a^m(\cos m\theta - \sin m\theta)}{r^{m-1}}$$

$$\frac{d\theta}{dr} = \frac{r^{m-1}}{a^m(\cos m\theta - \sin m\theta)}$$

$$r\frac{d\theta}{dr} = \frac{r^m}{a^m(\cos m\theta - \sin m\theta)} = \frac{a^m(\cos m\theta + \sin m\theta)}{a^m(\cos m\theta - \sin m\theta)}$$

$$= \frac{\cos m\theta(1 + \tan m\theta)}{\cos m\theta(1 - \tan m\theta)} = \frac{\tan(\pi/4) + \tan m\theta}{1 - \tan(\pi/4)\tan m\theta}$$

$$= \tan((\pi/4) + m\theta) \qquad \because 1 = \tan(\pi/4)$$

Therefore, $\tan\phi = r(d\theta/dr) = \tan((\pi/4) + m\theta)$

$$\Rightarrow \phi = (\pi/4) + m\theta$$ ■

4. *Find the angle between the radius vector and the tangent to the curve* $\frac{2a}{r} = 1 - \cos\theta$

➨ $\frac{2a}{r} = 1 - \cos\theta$

$$r(1 - \cos\theta) = 2a$$

$$r\sin\theta + (1-\cos\theta)\frac{dr}{d\theta} = 0$$

$$\frac{dr}{d\theta} = \frac{-r\sin\theta}{(1-\cos\theta)} \Rightarrow \frac{d\theta}{dr} = \frac{(1-\cos\theta)}{-r\sin\theta}$$

$$r\frac{d\theta}{dr} = -\frac{(1-\cos\theta)}{\sin\theta} = -\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\tan\left(\frac{\theta}{2}\right)$$

Therefore, $\tan\phi = r(d\theta/dr) = -\tan(\theta/2) = \tan(\pi-(\theta/2))$

$$\Rightarrow \phi = \pi - (\theta/2)$$ ■

5. *Find the angle between the radius vector and the tangent to the curve* $r^m \cos m\theta = a^m$

➡ $r^m \cos m\theta = a^m$

$$r^m(-m\sin m\theta) + \cos m\theta\, m r^{m-1}\frac{dr}{d\theta} = 0$$

$$\cos m\theta . m r^{m-1}\frac{dr}{d\theta} = m r^m \sin m\theta$$

$$\frac{dr}{d\theta} = \frac{r^m \sin m\theta}{r^{m-1}\cos m\theta} = r\tan m\theta$$

$$r\frac{d\theta}{dr} = \frac{1}{\tan m\theta} = \cot m\theta$$

Therefore, $\tan\phi = r(d\theta/dr) = \cot m\theta = \tan((\pi/2) - m\theta)$

$$\Rightarrow \phi = (\pi/2) - m\theta$$ ■

6. *Find the angle between the radius vector and the tangent to the curve* $r = e^{\theta\cot\alpha}$

➡ $r = e^{\theta\cot\alpha}$

$$\frac{dr}{d\theta} = e^{\theta\cot\alpha}\cot\alpha = r\cot\alpha$$

$$\frac{d\theta}{dr} = \frac{1}{r\cot\alpha}$$

$$r\frac{d\theta}{dr} = \frac{1}{\cot\alpha} = \tan\alpha$$

Therefore, $\tan\phi = r(d\theta/dr) = \tan\alpha$

$\Rightarrow \phi = \alpha$ ■

7. *Find the angle between the radius vector and the tangent and also find the slope of the tangent to the curve* $r^2 = a^2\sin 2\theta$ *at* $\theta = (\pi/12)$

➡ $r^2 = a^2\sin 2\theta$

$$2r\frac{dr}{d\theta} = a^2\cos 2\theta.2$$

$$\frac{dr}{d\theta} = \frac{a^2\cos 2\theta}{r}$$

$$\frac{d\theta}{dr} = \frac{r}{a^2\cos 2\theta}$$

$$r\frac{d\theta}{dr} = \frac{r^2}{a^2\cos 2\theta} = \frac{a^2\sin 2\theta}{a^2\cos 2\theta} = \tan 2\theta$$

Therefore, $\tan\phi = r(d\theta/dr) = \tan 2\theta$

$\Rightarrow \phi = 2\theta$

If $\theta = \frac{\pi}{12}$, then $\phi = 2\left(\frac{\pi}{12}\right) = \frac{\pi}{6}$

We have, $\psi = \theta + \phi = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi + 2\pi}{12} = \frac{\pi}{4}$

Therefore, the slope of the tangent $= \tan\psi = \tan(\pi/4) = 1$ ■

Exercises

I. *Find the angle between radius vector and the tangent for the following curves.*

1) $r\sec^2(\theta/2) = a$ 2) $r^2 = a^2(\sin 2\theta + \cos 2\theta)$

3) $\frac{l}{r} = 1 + e\cos\theta$ 4) $r = a(1 + \cos\theta)$

5) $r^n = a^n\sec(n\theta + \alpha)$ 6) $r^2\cos 2\theta = a^2$

7) $r^n = a^n \sin n\theta$ 8) $r = a(1+\sin\theta)$ at $\theta = (\pi/6)$

9) $r(1+\cos\theta) = 2a$

II. *Find the angle between the radius vector and the tangent and also find slope of the tangent for the following curves:*

1) $r = a(1+\cos\theta)$ *at* $\theta = (\pi/3)$

2) $r = a(1+\sin\theta)$ *at* $\theta = (\pi/2)$

Answers

I. 1) $(\pi+\theta)/2$ 2) $(\pi+8\theta)/4$ 3) $\tan^{-1}\left(\dfrac{1+e\cos\theta}{e\sin\theta}\right)$

4) $(\pi+\theta)/2$ 5) $(\pi/2)-(n\theta+\alpha)$ 6) $(\pi/2)-2\theta$

7) $n\theta$ 8) $\pi/2$ 9) $\theta/2$

II. 1) $2\pi/3$, 0 2) $\pi/2$, 0

Angle between two Polar Curves

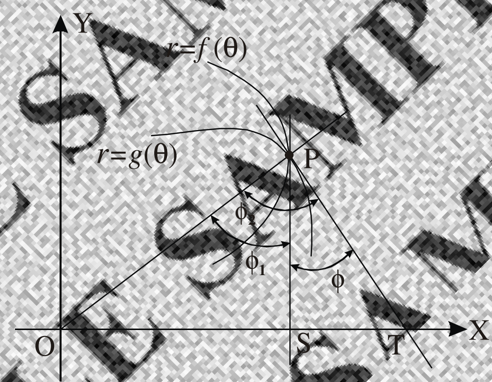

Figure 2.3

Suppose $r = f(\theta)$ and $r = g(\theta)$ are two Polar curves intersecting at the point P. Let PS and PT be the tangents to the curves $r = f(\theta)$ and $r = g(\theta)$ at the point P respectively. Then the angle between two curves is same as angle between the tangents PS and PT.

i.e., $\phi = |\phi_2 - \phi_1|$

or $\tan\phi = \tan|\phi_2 - \phi_1| \Rightarrow \tan\phi = |\tan(\phi_2 - \phi_1)|$

$$\tan\phi = \left|\frac{\tan\phi_2 - \tan\phi_1}{1+\tan\phi_2\tan\phi_1}\right|$$

Note Two curves are said to be orthogonal if $|\phi_2 - \phi_1| = \pi/2$ or $\tan\phi_1 \cdot \tan\phi_2 = -1$

Worked Examples

1. *Find the angle between the pairs of curves*
$r = a\cos\theta, \qquad 2r = a$

➡ $r = a\cos\theta \qquad 2r = a$

Differentiating both equations with respect to θ, we get

$$\frac{dr}{d\theta} = a(-\sin\theta) \qquad\qquad 2\frac{dr}{d\theta} = 0$$

$$\frac{d\theta}{dr} = -\frac{1}{a\sin\theta} \qquad\qquad \frac{dr}{d\theta} = 0$$

$$r\frac{d\theta}{dr} = -\frac{r}{a\sin\theta} = -\frac{a\cos\theta}{a\sin\theta} \qquad\qquad \frac{d\theta}{dr} = \infty$$

$$= -\cot\theta = \tan\left(\frac{\pi}{2}+\theta\right) \qquad\qquad r\frac{d\theta}{dr} = \infty$$

Therefore, $\tan\phi_1 = r(d\theta/dr) = \tan((\pi/2)+\theta)$

$$\Rightarrow \phi_1 = (\pi/2)+\theta$$

and $\tan\phi_2 = r(d\theta/dr) = \infty \Rightarrow \phi_2 = \pi/2$.

Hence, $\varphi = |\varphi_1 - \varphi_2| = \left|\frac{\pi}{2} - \frac{\pi}{2} - \theta\right| = \theta$ ---(1)

We have, $2r = a$ and $r = a\cos\theta$

Therefore, $2(a\cos\theta) = a \Rightarrow 2\cos\theta = 1$

$$\cos\theta = 1/2 \Rightarrow \theta = \pi/3$$

$$\phi = \theta = \pi/3$$ ■

2. *Find the angle between the pairs of curves*

$r = 6\cos\theta, \quad r = 2(1+\cos\theta)$

➡ $r = 6\cos\theta, \qquad r = 2(1+\cos\theta)$

Differentiating both equations with respect to θ, we get

$$\frac{dr}{d\theta} = 6(-\sin\theta) \qquad\qquad \frac{dr}{d\theta} = 2(-\sin\theta)$$

$$r\frac{d\theta}{dr} = -\frac{r}{6\sin\theta} \qquad\qquad r\frac{d\theta}{dr} = -\frac{r}{2\sin\theta}$$

$$= -\frac{6\cos\theta}{6\sin\theta} \qquad\qquad = -\frac{2(1+\cos\theta)}{2\sin\theta}$$

$$= -\cot\theta \qquad\qquad = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= \tan\left(\frac{\pi}{2}+\theta\right) \qquad\qquad = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

Therefore, $\tan\phi_1 = r(d\theta/dr) = \tan((\pi/2)+\theta)$

$$\Rightarrow \phi_1 = (\pi/2)+\theta$$

$$\tan\phi_2 = r(d\theta/dr) = \tan((\pi/2)+(\theta/2))$$

$$\Rightarrow \phi_2 = (\pi/2)+(\theta/2)$$

$$\phi = |\phi_2 - \phi_1| = \left|\frac{\pi}{2}+\frac{\theta}{2}-\frac{\pi}{2}-\theta\right| = \frac{\theta}{2} \qquad \text{---(1)}$$

We have,

$$r = 6\cos\theta, \qquad r = 2(1+\cos\theta)$$

$$\therefore 6\cos\theta = 2(1+\cos\theta)$$

$$6\cos\theta = 2+2\cos\theta$$

$$6\cos\theta - 2\cos\theta = 2$$

$$4\cos\theta = 2$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore (1) \Rightarrow \phi = \frac{\pi/3}{2} = \frac{\pi}{6}$$

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