(ii) Let $z = \frac{1-i}{1+i}\times\frac{1-i}{1-i} = \frac{(1-i)^2}{1^2-i^2} = \frac{1^2+i^2-2i}{2} = \frac{1-1-2i}{2}$

$z=-i$

$r=|z|=\sqrt{0^2+(-1)^2}=1$

$\alpha=\tan^{-1}\left|\frac{-1}{0}\right| = \tan^{-1}(\infty)=\frac{\pi}{2}$

Here $z=-i=(0,-1)$ is on negative y-axis.

Therefore, $\theta = \text{amp}\, z = \pi+\frac{\pi}{2}=\frac{3\pi}{2}$

Hence, exponential form of a given complex number is

$$z=re^{i\theta}=1\cdot e^{i\frac{3\pi}{2}}=e^{\frac{3\pi}{2}i}$$

(iii) Let $z=\frac{\sqrt{3}}{2}-\frac{1}{2}i$

$r=|z|=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$\alpha=\tan^{-1}\left|\frac{-(1/2)}{(\sqrt{3}/2)}\right|=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

Here, $z=\frac{\sqrt{3}}{2}-\frac{1}{2}i$ is in the 4th quadrant.

Therefore, $\theta=\text{amp}\, z = 2\pi-\alpha=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$

Hence, exponential form of a given complex number is

$$z=re^{i\theta}=1\cdot e^{i\frac{11\pi}{6}}=e^{\frac{11\pi}{6}i}$$

(iv) Let $z=\frac{(1+i)(1+2i)}{1+3i}=\frac{1+i+2i+2i^2}{1+3i}=\frac{1+3i-2}{1+3i}$

$$=\frac{-1+3i}{1+3i}\times\frac{1-3i}{1-3i}=\frac{-1+3i+3i-9i^2}{1-9i^2}=\frac{6i+8}{10}$$

$$z = \frac{4}{5} + \frac{3}{5}i$$

$$r = \sqrt{\frac{16}{5} + \frac{9}{25}} = 1$$

$$\alpha = \tan^{-1}\left|\frac{3/5}{4/5}\right| = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\theta = \alpha = \tan^{-1}\left(\frac{3}{4}\right)$$

Hence, exponential form of a given complex number is

$$z = re^{i\theta} = 1.e^{i\tan^{-1}\left(\frac{3}{4}\right)} = e^{i\tan^{-1}\left(\frac{3}{4}\right)}$$ ■

12. *Prove that* $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = i\cot\left(\dfrac{\theta}{2}\right)$

$$\Rightarrow \frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = \frac{2\cos^2\left(\frac{\theta}{2}\right)+i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)-i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \frac{2\cos^2\left(\frac{\theta}{2}\right)+2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2(-i^2)\sin^2\left(\frac{\theta}{2}\right)-2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \frac{2\cos\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right)}{-2i\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right)}$$

$$= -\frac{1}{i}\cot\left(\frac{\theta}{2}\right) = i\cot\left(\frac{\theta}{2}\right)$$

Therefore, $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = i\cot\left(\dfrac{\theta}{2}\right)$ ■

13. *Express the following in the form of* $x+iy$

(i) $\log i$ (ii) $\log(\log i)$

➡ (i) Since $i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=e^{i\frac{\pi}{2}}$

Therefore, $\log i=\log e^{i\frac{\pi}{2}}=i\frac{\pi}{2}\log_e e$

$$\log i=i\frac{\pi}{2} \quad \text{---(1)}$$

(ii) From equation (1), we have

$$\log i=i\frac{\pi}{2}$$

Therefore, $\log(\log i)=\log\left(i\frac{\pi}{2}\right)=\log i+\log\frac{\pi}{2}=i\frac{\pi}{2}+\log\frac{\pi}{2}$

$$\log(\log i)=\log\frac{\pi}{2}+i\frac{\pi}{2}$$ ■

14. *If* $\sqrt{a+ib}=x+iy$ *prove that* $\sqrt{a-ib}=x-iy$

➡ Here $\sqrt{a+ib}=x+iy$

$$a+ib=(x+iy)^2=x^2+2ixy+(iy)^2=x^2-y^2+i(2xy)$$

Equating real and imaginary parts, we get

$$a=x^2-y^2,\ b=2xy$$

$$a-ib=x^2-y^2-i2xy$$

$$a-ib=x^2+(iy)^2-2x(iy)=(x-iy)^2$$

$$\sqrt{a-ib}=(x-iy)$$ ■

Exercises

I. *Express the following numbers in the form* $x+iy$

(1) $\frac{1}{4+3i}$ (2) $\frac{1}{(2+i)^2}-\frac{1}{(2-i)^2}$

(3) $\dfrac{1}{1-\cos\theta+2i\sin\theta}$ (4) $\dfrac{1+i}{1-i}-\dfrac{1-i}{1+i}$

(5) $\dfrac{2+5i}{3-2i}+\dfrac{2-5i}{3+2i}$ (6) $\dfrac{(1-i)^3}{1-i^3}$

(7) $\dfrac{1+i}{1-i}$ (8) $\dfrac{\sqrt{3}+i}{\sqrt{3}-i}$

II. *Find the conjugate of the complex number.*

(1) $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ (2) $\dfrac{(1+i)^2}{3-i}$

(3) $\dfrac{(2-3i)(2+i)}{1+i}$ (4) $\dfrac{2+5i}{3-2i}+\dfrac{2-5i}{3+2i}$

(5) $\dfrac{3-i}{2+i}+\dfrac{3+i}{2-i}$ (6) $\dfrac{3}{1+i}-\dfrac{2}{2-i}+\dfrac{2}{1-i}$

(7) $\dfrac{1+i}{1-i}$ (8) $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$

III. *Find the real values x and y for which the following equations are satisfied.*

(1) $(1-i)x+(1+i)y=1$

(2) $(x+iy)(i-4)=8+15i$

(3) $\dfrac{(1+i)(x-2i)}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

(4) *If* $z^2=-4-3i$ *Find* z (*or find the value of* $\sqrt{-4-3i}$)

(5) *Find the real value of* θ *for which* $\dfrac{1+i\cos\theta}{1-2i\cos\theta}$ *is a real number.*

(6) *Find* z *if* $(x+iy)(2-3i)=4+i$ *where* $z=x+iy$

(7) *Evaluate* $\left(i^{19}+\dfrac{1}{i^{25}}\right)^2$

(8) *Evaluate* $\left(i^{17}-\frac{1}{i^{34}}\right)^3$

(9) *Find* $\sqrt{7-24i}$

(10) *Find the square root of* $8-15i$

(11) *If* $z^2+5=12\sqrt{-1}$ *find the complex number* z

(12) *Find all complex numbers satisfying* $\bar{z}=z^2$

(13) *Find the number of solutions to the equation* z [illegible] $z=0$

(14) *If* $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$ *then find a and b*

(15) *Prove that* $\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)^{50}=3^{25}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$

IV. *Express the following complex numbers in polar form and hence find their modulus and amplitude.*

(1) $\frac{1-i}{1+i}$ (2) $2i$

(3) -12 (4) $\frac{(2+i)^2}{3i}$

(5) $\left(\frac{2+i}{3-i}\right)^2$ (6) $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3$

(7) $-i\sqrt{2}-\sqrt{2}$ (8) $\frac{1+i}{2-(1-i)^2}$

(9) $\frac{z(z+2i)}{z+4i}$ *where* $z=1+i$

(10) $\frac{i+i^2-i^3+i^4+2}{i^5+i^6+2}$ (11) $4-4i\sqrt{3}$

Answers

I. (1) $\frac{4}{25}+\left(\frac{-3}{25}\right)i$ (2) $\frac{17}{2}-\frac{15}{2}i$

(3) $\frac{1}{(5+3\cos\theta)}-i\frac{2\cot\left(\frac{\theta}{2}\right)}{(5+3\cos\theta)}$ (4) 0+2i

(5) $-\frac{8}{13}+i0$ (6) $-2+i0$

(7) 0+i (8) $\frac{1}{2}+\left(\frac{\sqrt{3}}{2}\right)i$

II. (1) $\frac{63}{25}+i\frac{16}{25}$ (2) $-\frac{1}{5}-\frac{3}{5}i$

(3) $-\frac{17}{2}+\frac{15i}{2}$ (4) $-\frac{8}{13}-i0$

(5) 2 (6) $\frac{17}{10}+i\frac{9}{10}$

(7) $-i$ (8) $\frac{3}{2}i$

III. (1) $x=\frac{1}{2}$, $y=\frac{1}{2}$ (2) $x=-1, y=-4$

(3) $x=3, y=-1$ (4) $\frac{1}{\sqrt{2}}-\frac{3}{\sqrt{2}}i, -\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}}i$

(5) $\frac{\pi}{2}$ (6) $\frac{5}{13}+\frac{14}{13}i$

(7) -4 (8) $-2(1-i)$

(9) $\pm 4-3i$ (10) $\pm\frac{1}{\sqrt{2}}(5-3i)$

(11) $\pm(2+3i)$ (12) $0,1,\frac{-1}{2}+\frac{\sqrt{3}}{2}i,-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(13) 4 (14) $a=1,\ b=0$

IV. (1) $1,-\frac{\pi}{2}$ (2) $2,\frac{\pi}{2}$ (3) $12,\ \pi$

(4) $\frac{5}{4},\tan^{-1}\left(-\frac{3}{4}\right)$ (5) $\frac{1}{2},\frac{\pi}{2}$

(6) $2,-\frac{\pi}{2}$ (7) $2,\frac{-3\pi}{4}$ (8) $\frac{1}{2},0$

(9) $\sqrt{\frac{10}{13}},\tan^{-1}\frac{7}{9}$ (10) $2,0$

(11) $8,\frac{\pi}{3}$

De Moivre's Theorem

If n is any integer, then $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ and if n is a rational number say p/q, then $(\cos\theta+i\sin\theta)^{p/q}$ has q values and one of its values is $\cos(p/q)\,\theta+i\sin(p/q)\theta$

Proof

Case (1) Let n be a positive integer.

In this case we will prove the result by mathematical induction.

If $n=1$ then $(\cos\theta+i\sin\theta)^1=\cos 1\cdot\theta+i\sin 1\cdot\theta$

$$=\cos\theta+i\sin\theta.$$

Therefore, the result is true for $n=1$.

Let us assume that the result is true for $n=m$.

ie., $(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$ ---(1)

Multiplying both sides of (1) by $\cos\theta+i\sin\theta$, we get

$$(\cos\theta+i\sin\theta)^{m+1}=(\cos m\theta+i\sin m\theta)(\cos\theta+i\sin\theta)$$

$= (\cos m\theta \cos\theta - \sin m\theta \sin\theta) + i(\sin m\theta \cos\theta + \cos m\theta \sin\theta)$

$= \cos(m\theta + \theta) + i\sin(m\theta + \theta)$

$= \cos(m+1)\theta + i\sin(m+1)\theta$

Therefore, the result is true for $n = m + 1$
Hence, the result is true for all the value of n.

Case (2) Let n be a negative integer i.e., $n = -m$ where m is a positive integer. Consider $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta} \qquad \text{(from case (1))}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$= \cos m\theta - i\sin m\theta$

$= \cos(-m)\theta + i\sin(-m)\theta$

$= \cos n\theta + i\sin n\theta$

Case (3) Let n be a rational number i.e., $n = \frac{p}{q}$, where p and q are integers and $q \neq 0$

Let $z = \cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)$

Therefore, $z^q = \left[\cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)\right]^q$

$= \cos q\left(\frac{p}{q}\theta\right) + i\sin q\left(\frac{p}{q}\theta\right)$

$=$ $\cos p\theta + i\sin p\theta$

$=$ $(\cos\theta + i\sin\theta)^p$ (From case (1) and (2))

$$z^q = (\cos\theta + i\sin\theta)^P$$

Taking q^{th} root on both sides we get q values of z and one of these q values is equal to $(\cos\theta + i\sin\theta)^{\frac{P}{q}}$

Therefore, $z = (\cos\theta + i\sin\theta)^{\frac{P}{q}}$

i.e., $\cos\left(\frac{P}{q}\theta\right) + i\sin\left(\frac{P}{q}\theta\right) = (\cos\theta + i\sin\theta)^{\frac{P}{q}}$

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is true for all the values of n

i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Note (1) By replacing $-\theta$ in (1), we get

$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

(2) $\cos\theta + i\sin\theta = \text{cis}\,\theta$

(3) $\cos\theta - i\sin\theta = \text{cis}(-\theta)$

Prove the following

(i) $\text{cis}\,\alpha\ \text{cis}\,\beta = \text{cis}(\alpha+\beta)$ (ii) $\frac{\text{cis}\,\alpha}{\text{cis}\,\beta} = \text{cis}(\alpha-\beta)$

Proof (i) $\text{cis}\,\alpha\ \text{cis}\,\beta = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= \cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta + i^2\sin\alpha\sin\beta$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

$\text{cis}\,\alpha\,\text{cis}\,\beta = \text{cis}(\alpha+\beta)$

(ii) $\frac{\text{cis}\,\alpha}{\text{cis}\,\beta} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} \times \frac{\cos\beta - i\sin\beta}{\cos\beta - i\sin\beta}$

$$= \frac{\cos\alpha\cos\beta + i\sin\alpha\cos\beta - i\cos\alpha\sin\beta - i^2\sin\alpha\sin\beta}{\cos^2\beta - i^2\sin^2\beta}$$

$$= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta + i[\sin\alpha\cos\beta - \cos\alpha\sin\beta]}{\cos^2\beta + \sin^2\beta}$$

$$= \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

$$\frac{\text{cis}\,\alpha}{\text{cis}\,\beta} = \text{cis}(\alpha-\beta)$$

Similarly

$$\text{cis}\,\alpha_1 \cdot \text{cis}\,\alpha_2 \cdot \text{cis}\,\alpha_3 \ldots\ldots\ldots \text{cis}\,\alpha_n = \text{cis}\,(\alpha_1+\alpha_2+\alpha_3 \ldots\ldots\ldots \alpha_n)$$

Worked Examples

1. *Simplify* $\dfrac{(\cos 6\theta - i\sin 6\theta)^3 (\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta - i\sin 4\theta)^3}$

➡ $$\frac{(\cos 6\theta - i\sin 6\theta)^3 (\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta - i\sin 4\theta)^3} = \frac{(\text{cis}(-6\theta))^3 (\text{cis}\,2\theta)^7}{[\text{cis}(-4\theta)]^3}$$

$$\frac{(\text{cis}\,\theta)^{-6\times 3} \times (\text{cis}\,\theta)^{2\times 7}}{(\text{cis}\,\theta)^{-4\times 3}} = (\text{cis}\,\theta)^{-18+14+12}$$

$$= (\text{cis}\,\theta)^8 = \text{cis}\,8\theta = \cos 8\theta + i\sin 8\theta$$

2. *Prove that* $\left(-1+i\sqrt{3}\right)^{3n} + \left(-1-i\sqrt{3}\right)^{3n} = 2^{3n+1}$, *where n is an integer.*

➡ We have, $-1+i\sqrt{3} = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

$$\left(-1+i\sqrt{3}\right)^{3n} = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{3n}$$

$$= 2^{3n}\left[\cos 2n\pi + i\sin 2n\pi\right] \qquad \text{---(1)}$$

Similarly $-1-i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

$$\Rightarrow \left(-1-i\sqrt{3}\right)^{3n} = 2^{3n}\left[\cos 2n\pi - i\sin 2n\pi\right] \qquad \text{---(2)}$$

Adding equation (1) and (2), we get

$$\left(-1+i\sqrt{3}\right)^{3n} + \left(-1-i\sqrt{3}\right)^{3n} = 2.2^{3n}\cos 2n\pi$$

$$= 2^{3n+1} \qquad \because \cos 2n\pi = (-1)^{2n} = 1 \quad ■$$

3. *If α and β are the roots of the equation*

$x^2-2x+4=0$, *then prove that* $\alpha^n+\beta^n=2^{n+1}\cos\frac{n\pi}{3}$

➡ $x^2-2x+4=0$

$$x=\frac{2\pm\sqrt{4-4(1)(4)}}{2(1)}=\frac{2\pm\sqrt{-12}}{2}=\frac{2\pm 2i\sqrt{3}}{2}$$

$$=1\pm i\sqrt{3}=\alpha,\ \beta$$

Therefore, $\alpha=1+i\sqrt{3},\ \beta=1-i\sqrt{3}$

We have,

$$\alpha=1+i\sqrt{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\alpha^n=\left(1+i\sqrt{3}\right)^n=2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n$$

$$\alpha^n=2^n\left[\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right] \qquad \text{---(1)}$$

Similarly $\beta=1-i\sqrt{3}=2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$

$$\therefore\ \beta^n=\left(1-i\sqrt{3}\right)^n=2^n\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n$$

$$\beta^n=2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right) \qquad \text{---(2)}$$

$$\therefore\ \alpha^n+\beta^n=2.2^n\cos\frac{n\pi}{3}=2^{n+1}\cos\frac{n\pi}{3}$$ ■

4. *Prove that*

$$(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n=2^{n+1}\cos\frac{n\theta}{2}\cos\frac{n\theta}{2}$$

➡ $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n$

$$=\left[2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n+\left[\cos^2\left(\frac{\theta}{2}\right)-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n$$

$$= 2^n \cos^n \frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]^n + 2^n \cos^n \frac{\theta}{2}\left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right]^n$$

$$= 2^n \cos^n \frac{\theta}{2}\left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right] + 2^n \cos^n \frac{\theta}{2}\left[\cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right]$$

$$= 2^n \cos^n \frac{\theta}{2}\left[2\cos\frac{n\theta}{2}\right]$$

$$= 2^{n+1} \cos^n \frac{\theta}{2} \cos\frac{n\theta}{2}$$ ■

5. *Show that* $(a+ib)^n + (a-ib)^n = 2\left(a^2+b^2\right)^{\frac{n}{2}} \cos\left(n\tan^{-1}\left(\frac{b}{a}\right)\right)$

➡ If $a = r\cos\theta$ and $b = r\sin\theta$ then

$$a^2 + b^2 = r^2 \Rightarrow r = \left(a^2+b^2\right)^{\frac{1}{2}}$$

and $\frac{b}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$

Therefore, $(a+ib) = r(\cos\theta + i\sin\theta)$

$$(a+ib)^n = r^n(\cos n\theta + i\sin n\theta) \quad \text{---(1)}$$

and $(a-ib) = r(\cos\theta - i\sin\theta)$

$$(a-ib)^n = r^n(\cos n\theta - i\sin n\theta) \quad \text{---(2)}$$

Adding equations (1) and (2), we get

$$(a+ib)^n + (a-ib)^n = 2r^n \cos n\,\theta$$

$$(a+ib)^n + (a-ib)^n = 2\left(a^2+b^2\right)^{\frac{n}{2}} \cos\left(n\tan^{-1}\left(\frac{b}{a}\right)\right)$$ ■

6. *If* $x + \frac{1}{x} = 2\cos\alpha,\ y + \frac{1}{y} = 2\cos\beta$ *and* $z + \frac{1}{z} = 2\cos\gamma$, *then prove the following*

(i) $x^n + \frac{1}{x^n} = 2\cos n\alpha$ (ii) $x^n - \frac{1}{x^n} = 2i\sin n\alpha$

(iii) $\frac{(x^n)}{(y^m)} + \frac{(y^m)}{(x^n)} = 2\cos(n\alpha - m\beta)$

(iv) $\frac{(x^n y^m)}{z^l} + \frac{z^l}{x^n y^m} = 2\cos(n\alpha + m\beta - l\gamma)$

➡ $x + \frac{1}{x} = 2\cos\alpha$

$x^2 - 2x\cos\alpha + 1 = 0$

$x = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4(1)(1)}}{2(1)}$

$= \frac{2\cos^2\alpha \pm \sqrt{4(\cos^2\alpha - 1)}}{2} = \frac{2\cos^2\alpha \pm 2\sqrt{-\sin^2\alpha}}{2}$

$x = \cos\alpha \pm i\sin\alpha$

Let $x = \cos\alpha + i\sin\alpha = \text{cis}\,\alpha$

Similarly $y = \cos\beta + i\sin\beta = \text{cis}\,\beta$

$z = \cos\gamma + i\sin\gamma = \text{cis}\,\gamma$

(i) $x^n = (\text{cis}\,\alpha)^n = \text{cis}\,n\alpha$

$x^n = \cos n\alpha + i\sin n\alpha$ ---(1)

and $\frac{1}{x^n} = \cos n\alpha - i\sin n\alpha$ ---(2)

Adding (1) and (2), we get

$\therefore x^n + \frac{1}{x^n} = 2\cos n\alpha$

(ii) Subtracting (2) from (1), we get

$x^n - \frac{1}{x^n} = 2i\sin n\alpha$