$$\log(1+x)=\log\left[\frac{c_1 y}{1-y}\right]$$

$$(1+x)=\frac{c_1 y}{1-y}$$

$$(1+x)(1-y)=c_1 y \qquad \text{---(2)}$$

Given $y=2$ at $x=1$, therefore, from (2), we get

$$(1+1)(1-2)=c_1(2)$$

$$-2=2c_1 \Rightarrow c_1=-1$$

(2) becomes, $(1+x)(1-y)=-y$ ■

11. *Solve* $\frac{dy}{dx}=16+4x^2+4y^2+x^2y^2$

➡ $\frac{dy}{dx}=16+4x^2+4y^2+x^2y^2=4(4+x^2)+y^2(4+x^2)$

$$\frac{dy}{dx}=(4+y^2)(4+x^2)$$

$$\frac{dy}{4+y^2}=(4+x^2)dx$$

$$\int\frac{1}{4+y^2}dy=\int(4+x^2)dx+c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)=4x+\frac{x^3}{3}+c$$ ■

Equations Reducible to Separable Form

If the differential equation can be put in the form

$$\frac{dy}{dx}=f(ax+by+c) \qquad \text{---(1)}$$

Then, it can be reduced to separation of variables, by putting $ax+by+c=z$

Therefore, $a+b\frac{dy}{dx}=\frac{dz}{dx}$

$$\frac{dy}{dx}=\frac{1}{b}\left(\frac{dz}{dx}-a\right)$$

(1) becomes, $\frac{1}{b}\left(\frac{dz}{dx}-a\right)=f(z)$

which is variable separable form.

12. *Solve* $\frac{dy}{dx}=(4x+y+1)^2$

➡ $\frac{dy}{dx}=(4x+y+1)^2$ ---(1)

Put $4x+y+1=t$

$$4+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{dt}{dx}-4$$

(1) becomes, $\frac{dt}{dx}-4=t^2$

$$\frac{dt}{dx}=t^2+4$$

$$\frac{dt}{t^2+4}=dx$$

$$\int\frac{dt}{t^2+4}=\int dx+c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)=x+c$$

$$\frac{1}{2}\tan^{-1}\left[\frac{4x+y+1}{2}\right]=x+c$$ ■

13. *Solve* $\frac{dy}{dx}=\sin(x+y+1)+\cos(x+y+1)$

➡ $\frac{dy}{dx}=\sin(x+y+1)+\cos(x+y+1)$ ---(1)

Put $x+y+1=z$

$$1+\frac{dy}{dx}=\frac{dz}{dx}$$

$$\frac{dy}{dx}=\frac{dz}{dx}-1$$

(1) becomes, $\frac{dz}{dx}-1=\sin z+\cos z$

$$\therefore \frac{dz}{dx}=\sin z+\cos z+1$$

$$\frac{dz}{\sin z+\cos z+1}=dx$$

$$\int \frac{dz}{\sin z+\cos z+1}=\int dx+c$$

Put $\tan\frac{z}{2}=t \Rightarrow dz=\frac{2dt}{1+t^2}$, $\sin z=\frac{2t}{1+t^2}$, $\cos z=\frac{1-t^2}{1+t^2}$

$$\therefore \int \frac{1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+1}\,\frac{2dt}{1+t^2}=x+c$$

$$\int \frac{1}{\frac{2t+1-t^2+(1+t^2)}{1+t^2}}\,\frac{2dt}{1+t^2}=x+c$$

$$2\int \frac{dt}{2t+2}=x+c \Rightarrow 2\int \frac{dt}{2(t+1)}=x+c$$

$$\log(t+1)=x+c$$

$$\log\left[\tan\left(\frac{z}{2}\right)+1\right]=x+c$$

$$\log\left[\tan\left(\frac{x+y+1}{2}\right)+1\right]=x+c$$ ■

14. *Solve* $\frac{dy}{dx}=\cos(x+y)$

➡ $\frac{dy}{dx}=\cos(x+y)$ ---(1)

Put $x+y=t \Rightarrow 1+\frac{dy}{dx}=\frac{dt}{dx} \Rightarrow \frac{dy}{dx}=\frac{dt}{dx}-1$

(1) becomes, $\frac{dt}{dx}-1=\cos t \Rightarrow \frac{dt}{dx}=\cos t+1$

$$\frac{dt}{\cos t+1}=dx$$

$$\int\frac{dt}{\cos t+1}=\int dx+c$$

$$\int\frac{1-\cos t}{(1-\cos t)(1+\cos t)}dt=x+c$$

$$\int\frac{(1-\cos t)dt}{1-\cos^2 t}=x+c$$

$$\int\frac{1-\cos t}{\sin^2 t}dt=x+c$$

$$\int\left(\frac{1}{\sin^2 t}-\frac{\cos t}{\sin^2 t}\right)dt=x+c$$

$$\int\left(\operatorname{cosec}^2 t-\operatorname{cosec} t\cot t\right)dt=x+c$$

$$-\cot t-(-\operatorname{cosec} t)=x+c$$

$$\operatorname{cosec}(x+y)-\cot(x+y)=x+c$$ ■

15. *Solve* $\frac{dy}{dx}+x\tan(y-x)=1$

➡ $\frac{dy}{dx}+x\tan(y-x)=1$ ---(1)

Put $y-x=t \Rightarrow \frac{dy}{dx}-1=\frac{dt}{dx} \Rightarrow \frac{dy}{dx}=\frac{dt}{dx}+1$

(1) becomes, $\frac{dt}{dx}+1+x\tan t=1$

$$\frac{dt}{dx}+x\tan t=0$$

$$\frac{dt}{dx}=-x\tan t$$

$$\frac{dt}{\tan t}=-xdx$$

$$\int \cot t\ dt=-\int xdx+c$$

$$\log \sin t=-\frac{x^2}{2}+c$$

$$\log[\sin(y-x)]=-\frac{x^2}{2}+c$$

$$\log[\sin(y-x)]+\frac{x^2}{2}=c$$ ■

16. *Solve* $\tan y\ y'=\sin(x+y)+\sin(x-y)$

➡ $\tan y\ y'=\sin(x+y)+\sin(x-y)$

$$\tan y\frac{dy}{dx}=\sin x\cos y+\cos x\sin y+\sin x\cos y-\cos x\sin y$$

$$\tan y\frac{dy}{dx}=2\sin x\cos y$$

$$\frac{\tan y}{\cos y}dy=2\sin x\,dx,$$

$$\int \tan y\sec y\ dy=2\int \sin xdx+c$$

$$\sec y=-2\cos x+c$$

$$\sec y+2\cos x=c$$ ■

17. *Solve* $\frac{dy}{dx}=3(1-x+y)+(x-y)^2$

➡ $\frac{dy}{dx}=3(1-x+y)+(x-y)^2$

$$\frac{dy}{dx}=3[1-(x-y)]+(x-y)^2 \qquad \text{---(1)}$$

Put $x-y=t$

$$1-\frac{dy}{dx}=\frac{dt}{dx}$$

$$1-\frac{dt}{dx}=\frac{dy}{dx}$$

(1) becomes, $1-\frac{dt}{dx}=3[1-t]+t^2$

$$1-\frac{dt}{dx}=3-3t+t^2$$

$$1-3+3t-t^2=\frac{dt}{dx}$$

$$-\left(t^2-3t+2\right)=\frac{dt}{dx}$$

$$-dx=\frac{dt}{t^2-3t+2}$$

$$\int dx+\int\frac{dt}{t^2-3t+2}=c \qquad \text{---(2)}$$

Let $\frac{1}{t^2-3t+2}=\frac{1}{(t-1)(t-2)}=\frac{A}{t-1}+\frac{B}{t-2}$

$$1=A(t-2)+B(t-1)$$

Put $t=1\Rightarrow A=-1$

$t=2\Rightarrow B=1$

Therefore, $\frac{1}{t^2-3t+2}=\frac{-1}{t-1}+\frac{1}{t-2}$

(2) becomes, $\int dx+\int\left(-\frac{1}{t-1}+\frac{1}{t-2}\right)dt=c$

$$x-\log(t-1)+\log(t-2)=c$$

$$x+\log\left(\frac{t-2}{t-1}\right)=c$$

$$x+\log\left[\frac{x-y-2}{x-y-1}\right]=c$$

■

Exercises

Solve the following equations

1) $\cos x\dfrac{dy}{dx} + \cos 2x = \cos 3x$

2) $x\cos y\,dy = \left(xe^x \log x + e^x\right)dx$

3) $\dfrac{dy}{dx} + xy = xy^2$

4) $\dfrac{\sec^2 x}{\tan x}dx + \dfrac{\sec^2 y}{\tan y}dy = 0$

5) $\dfrac{dy}{dx} = e^{x+y} + x^2 e^y$

6) $\dfrac{dy}{dx} = (2\operatorname{cosec} 2x + \cot x)\,y$

7) $\dfrac{dy}{dx} = \dfrac{e^x\left(\sin^2 x + \sin 2x\right)}{y(2\log y + 1)}$

8) $\dfrac{dy}{dx} + \tan x \sec x \cos^2 y = 0$

9) $\left(e^y + 1\right)\cos x\,dx + e^y \sin x\,dy = 0$

10) $\log\left(\dfrac{dy}{dx}\right) = ax + b$

11) $\dfrac{dy}{dx} = \sin(x + y)$

12) $\cos(x + y)dy = dx$

13) $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$

14) $2x^2 y\dfrac{dy}{dx} = \tan\left(x^2 y^2\right) - 2xy^2$ [Hint: Put $x^2 y^2 = z$]

15) $y\dfrac{dy}{dx} + x^2 = 0$

16) $x\dfrac{dy}{dx} = y\log x$

17) $\dfrac{dy}{dx} = (x + y)^2$

18) $x\dfrac{dy}{dx} = \pm ay$

19) $x\dfrac{dy}{dx} + \left(2 - ax^2\right)y = 0$

20) $x\dfrac{dy}{dx} = y\log y$

21) $3y' + 5\cot x \cot y \cos^2 y = 0$

22) $\dfrac{dy}{dx} = (3 + x - 4y)^2$

23) $\dfrac{dy}{dx} = (1 + 4x + 9y)^2$

24) Solve $(x + y + 1)^2 \dfrac{dy}{dx} = 1$

25) Solve $\dfrac{dy}{dx} = (3x + 2y + 4)^2$

26) Solve $\dfrac{dy}{dx} = 1 + 6xe^{x-y}$

27) $y\dfrac{dy}{dx} = x\sqrt{1+x^2+y^2+x^2y^2}$

28) $x^2y^2\dfrac{dy}{dx} = 1+x+y+xy$

29) $\dfrac{dy}{dx}\left[(x+y+1)^2+(x+y+1)+5\right]=1$

30) $(xy^2+x)dx+(x^2y+y)dy=0$

Answers

1) $y = \sin 2x - 2\sin x - x + \log(\sec x + \tan x) + c$

2) $\sin y = e^x \log x + c$

3) $\log\dfrac{y-1}{y} = \dfrac{x^2}{2} + c$

4) $\tan x \tan y = c$

5) $e^x + e^{-y} + \dfrac{x^3}{3} = c$

6) $y = c\sin x \tan x$

7) $y^2 \log y = e^x \sin^2 x + c$

8) $\sec x + \tan y = c$

9) $\sin x(e^y + 1) = c$

10) $ae^{-bx} + be^{ax} = c$

11) $(x+c)\left[1+\tan\left\{\dfrac{(x+y)}{2}\right\}\right]+2=0$

12) $y+c=\tan\left[\dfrac{(x+y)}{2}\right]$

13) $\log\left[\tan\left(\dfrac{x+y}{2}\right)\right]=x+c$

14) $\sin(x^2y^2)=ce^x$

15) $y^2 = -\dfrac{2}{3}x^3 + c$

16) $y = ce^{\frac{1}{2}(\log x)^2}$

17) $(x+y)=\tan(x+c)$

18) $y = cx^{\pm a}$

19) $x^2y = ce^{ax^2/2}$

20) $\log y = cx$

21) $3\sec^2 y + 10\log\sin x = c$

22) $\log\left(\dfrac{7+2x-8y}{5+2x-8y}\right)=c+4x$

23) $\tan^{-1}\left[\dfrac{3}{2}(1+4x+9y)\right]=6x+c$

24) $y=\tan^{-1}(x+y+1)+c$

25) $\dfrac{2}{\sqrt{6}}\tan^{-1}\left[\sqrt{\dfrac{2}{3}}(3x+2y+4)\right]=x+c$

26) $e^{x-y} = 3x^2 + c$

27) $\sqrt{1+y^2} - \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} = c$

28) $\frac{y^2}{2} - y + \frac{1}{x} + \log\left(\frac{1+y}{x}\right) = c$

29) $y + 1 - \frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{2x+2y+3}{\sqrt{23}}\right) = c$

30) $(1+x)^2(1+y)^2 = c$

HOMOGENEOUS EQUATIONS

The differential equation $dy/dx = f(x, y)$ is said to be *homogeneous equation*, if $f(x, y)$ can be expressed as a function of y/x or function of x/y.

Working rule

If $f(x, y)$ is a function of $\frac{y}{x}$ i.e., $\frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right)$

Put $\frac{y}{x} = v$ i.e., $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute the above substitution in the given differential equation, we get

$$v + x\frac{dv}{dx} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

which is a variable separable form.

Note **(i)** *In the differential equation* $f(x, y)dx + g(x, y)dy = 0$, *if* $f(x, y)$ *and* $g(x, y)$ *are homogeneous functions of x and y of the same degree, then the equation is called Homogeneous equation.*