Substituting (2) and (3) in (1), we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{e^{in\theta}}{r^n} - \frac{e^{-in\theta}}{r^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\left[\frac{e^{in\theta} - e^{-in\theta}}{2i}\right]$$

$$D^n\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

(3) Let $y = \tanh^{-1}\left(\frac{x}{a}\right)$

Differentiating with respect to x, we get

$$y_1 = \frac{1}{1-(x/a)^2}\frac{1}{a} = \frac{1}{a}\frac{1}{\left(a^2 - x^2\right)/a^2}$$

$$= \frac{a}{a^2 - x^2} = \frac{a}{(a-x)(a+x)} \qquad \text{---(1)}$$

Let $\frac{a}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x} = \frac{A(a+x)+B(a-x)}{(a-x)(a+x)}$

$$a = A(a+x) + B(a-x)$$

Put $x = a$ $\quad A = 1/2$

Put $x = -a$ $\quad B = 1/2$

$$\therefore \frac{a}{(a-x)(a+x)} = \frac{1/2}{a-x} + \frac{1/2}{a+x} \qquad \text{---(2)}$$

Substituting (2) in (1), we get

$$y_1 = \frac{1/2}{a-x} + \frac{1/2}{a+x}$$

$$y_1 = \frac{1}{2}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$$

Differentiating (n-1) times, we get

$$D^{n-1}(y_1) = \frac{1}{2}D^{n-1}\left[\frac{1}{(a-x)} + \frac{1}{(a+x)}\right]$$

$$y_n = \frac{1}{2}\left\{\frac{(-1)^{n-1}(n-1)!(-1)^{n-1}}{(a-x)^n} + \frac{(-1)^{n-1}(n-1)!}{(a+x)^n}\right\}$$

$$= \frac{1}{2}\left\{\frac{(n-1)!}{(a-x)^n} + \frac{(-1)^{n-1}(n-1)!}{(a+x)^n}\right\} = \frac{(n-1)!}{2}\left[\frac{1}{(a-x)^n} + \frac{(-1)^{n-1}}{(a+x)^n}\right] \quad ■$$

V. *Find the n^{th} derivatives of the following*

(1) $\tan^{-1} x$ (2) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(3) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (4) $\tan^{-1}\left(\frac{bx-a}{ax+b}\right)$

➡ (1) Let $y = \tan^{-1} x$

$$y_1 = \frac{1}{1+x^2} = \frac{1}{x^2 - i^2} = \frac{1}{(x+i)(x-i)} \qquad \text{---(1)}$$

Let $\frac{1}{(x+i)(x-i)} = \frac{A}{x+i} + \frac{B}{x-i} = \frac{A(x-i)+B(x+i)}{(x+i)(x-i)}$

$$1 = A(x-i) + B(x+i)$$

Put $x = i \Rightarrow B = 1/2i$

Put $x = -i \Rightarrow A = -1/2i$

$$\therefore \frac{1}{(x+i)(x-i)} = \frac{-1/2i}{x+i} + \frac{1/2i}{x-i} = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right] \qquad \text{---(2)}$$

Substituting equation (2) in (1), we get

$$y_1 = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

Differentiating (*n-1*), we get

$$D^{n-1}(y_1) = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right] \qquad \text{---(3)}$$

Put $x = r\cos\theta \quad 1 = r\sin\theta$

$x + i = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$

$$(x+i) = re^{i\theta}$$

$$(x+i)^n = r^n e^{in\theta}$$

$$\therefore \frac{1}{(x+i)^n} = \frac{1}{r^n e^{in\theta}} = \frac{e^{-in\theta}}{r^n} \qquad \text{---(4)}$$

Similarly, $\dfrac{1}{(x-i)^n} = \dfrac{e^{in\theta}}{r^n}$ ---(5)

Substituting (4) and (5) in (3), we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{e^{in\theta}}{r^n} - \frac{e^{-in\theta}}{r^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\left[\frac{e^{in\theta} - e^{-in\theta}}{2i}\right]$$

$$D^n\left(\tan^{-1} x\right) = \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta,$$

where $r = \sqrt{x^2+1}$, $\theta = \tan^{-1}(1/x)$

(2) Let $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$

$$\therefore y = \cos^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$= \cos^{-1}\left[\frac{1-\tan^2\theta}{\sec^2\theta}\right] = \cos^{-1}\left[\frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\right]$$

$$= \cos^{-1}\left[\cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta\right] = \cos^{-1}\left[\cos^2\theta - \sin^2\theta\right]$$

$$= \cos^{-1}(\cos 2\theta) \qquad \left[\because \cos^2\theta - \sin^2\theta = \cos 2\theta\right]$$

$$y = 2\theta$$

$$y = 2\tan^{-1} x$$

$$\therefore y_n = 2D^n\left(\tan^{-1} x\right)$$

This is same as **Example 1** and, we have

$$\therefore y_n = 2\frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

where $r = \sqrt{x^2+1}, \quad \theta = \tan^{-1}(1/x)$

(3) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

$$y = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= \sin^{-1}\left[\frac{2\tan\theta}{\sec^2\theta}\right] = \sin^{-1}\left[2\tan\theta\cdot\cos^2\theta\right]$$

$$= \sin^{-1}\left[2\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta\right]$$

$$= \sin^{-1}\left[2\sin\theta\cos\theta\right] \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta$$

$$y = 2\tan^{-1} x$$

$$y_n = 2D^n\left(\tan^{-1} x\right)$$

This is same as **Example 1** and, we have

$$y_n = 2\frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta,$$

where $r = \sqrt{x^2+1}, \quad \theta = \tan^{-1}(1/x)$

(4) Let $y = \tan^{-1}\left(\frac{bx-a}{ax+b}\right)$

$$= \tan^{-1}\left[\frac{b(x-(a/b))}{b((a/b)x+1)}\right] = \tan^{-1}\left[\frac{x-(a/b)}{1+(a/b)x}\right]$$

Put $x = \tan\theta$ and $(a/b) = \tan\alpha$

$$\therefore y = \tan^{-1}\left[\frac{\tan\theta - \tan\alpha}{1 + \tan\alpha\tan\theta}\right]$$

$$y = \tan^{-1}\left[\tan(\theta - \alpha)\right] = \theta - \alpha$$

$$y = \tan^{-1} x - \tan^{-1}(a/b)$$

$$\therefore y_n = D^n\left(\tan^{-1} x\right) - D^n\left(\tan^{-1}(a/b)\right)$$

$$y_n = D^n\left(\tan^{-1} x\right) - 0$$

This is same as **Example 1** and, we have

$$y_n = \frac{(-1)^{n-1}(n-1)\,!}{r^n}\sin n\theta,$$

$$\text{where } r^n = \sqrt{x^2+1}, \quad \theta = \tan^{-1}(1/x)$$ ■

Exercises

Find the n^{th} derivatives of the following functions

1) e^{ax+b}
2) $\sin 2x \sin 3x$
3) $(ax+b)^p$
4) $\cos^2 x \sin^3 x$
5) $e^{ax}\sin bx \cos cx$
6) $\cos^4 x$
7) $e^x \sin^4 x$
8) $e^{2x}\cos x \sin^2 2x$
9) $\sin 5x \sin 3x$
10) $\sin x \sin 2x \sin 3x$
11) $\cos x \cos 2x \cos 3x$
12) $\log\left[(3x+4)e^{\sin(5x+6)}\right]$
13) $\log\dfrac{3x-5}{2x+3}$
14) $\log\left(x^2-4\right)$
15) $\log\sqrt{\dfrac{x+2}{3x-1}}$
16) $\left(a^2-x^2\right)^{-1}$
17) $\dfrac{x^2}{(x+1)^2(x+2)}$
18) $\dfrac{x^2}{2x^2+7x+6}$
19) $\dfrac{x^4}{(x+1)(x+2)}$
20) $\dfrac{1}{1-x-x^2-x^3}$
21) $\dfrac{x^3}{x^2-1}$
22) $\dfrac{x}{x^2+a^2}$

23) $\dfrac{ax+b}{cx+d}$

24) $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$ [Hint put $x=\tan\theta$]

25) $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ [Hint put $x=\tan\theta$]

26) $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$ [Hint put $x=\tan\theta$, $1=\tan\dfrac{\pi}{4}$]

27) $\tan^{-1}\left(\dfrac{a-x}{1+ax}\right)$ [Hint put $x=\tan\theta$, $a=\tan\alpha$]

28) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

Answers

1) $a^n e^{ax+b}$

2) $\dfrac{1}{2}\left\{\cos\left(x+\dfrac{n\pi}{2}\right)-5^n\cos\left(5x+\dfrac{n\pi}{2}\right)\right\}$

3) $\dfrac{a^n p!(ax+b)^{p-n}}{(p-n)!}$

4) $\dfrac{1}{16}\left\{2\sin\left(x+\dfrac{n\pi}{2}\right)+3^n\sin\left(3x+\dfrac{n\pi}{2}\right)-5^n\sin\left(5x+\dfrac{n\pi}{2}\right)\right\}$

5) $\dfrac{1}{2}r^n e^{ax}\sin\{(b+c)x+n\phi\}+\dfrac{1}{2}r'^n e^{ax}\sin\{(b-c)x+n\psi\}$

where, $r^2=a^2+(b+c)^2, \phi=\tan^{-1}\left(\dfrac{b+c}{a}\right)$

$r'^2=a^2+(b-c)^2, \psi=\tan^{-1}\left(\dfrac{b-c}{a}\right)$

6) $\dfrac{1}{8}\left\{4^n\cos\left(4x+\dfrac{n\pi}{2}\right)+2^{n+2}\cos\left(2x+\dfrac{n\pi}{2}\right)\right\}$

7) $\frac{1}{8}e^{x}\left[3-4.5^{\frac{n}{2}}\cos\left(2x+n\tan^{-1}2\right)-17^{\frac{n}{2}}\cos\left(4x+n\tan^{-1}4\right)\right]$

8) $\frac{1}{4}e^{2x}\left[25^{\frac{n}{2}}\cos\left(x+n\tan^{-1}\frac{1}{2}\right)-13^{\frac{n}{2}}\cos\left(3x+n\tan^{-1}\frac{3}{2}\right)\right.$

$\left.-29^{\frac{n}{2}}\cos\left(5x+n\tan^{-1}\frac{5}{2}\right)\right]$

9) $\frac{1}{2}\left[2^{n}\cos\left(2x+\frac{n\pi}{2}\right)-8^{n}\cos\left(8x+\frac{n\pi}{2}\right)\right]$

10) $\frac{1}{4}\left[2^{n}\sin\left(2x+\frac{n\pi}{2}\right)-6^{n}\sin\left(6x+\frac{n\pi}{2}\right)+4^{n}\sin\left(4x+\frac{n\pi}{2}\right)\right]$

11) $\frac{1}{4}\left[2^{n}\cos\left(2x+\frac{n\pi}{2}\right)+4^{n}\cos\left(4x+\frac{n\pi}{2}\right)+6^{n}\cos\left(6x+\frac{n\pi}{2}\right)\right]$

12) $\frac{(-1)^{n-1}(n-1)!3^{n}}{(3x+4)^{n}}+5^{n}\sin\left(5x+6+\frac{n\pi}{2}\right)$

13) $(-1)^{n-1}(n-1)!\left[\frac{3^{n}}{(3x-5)^{n}}-\frac{2^{n}}{(2x+3)^{n}}\right]$

14) $(-1)^{n-1}(n-1)!\left[\frac{1}{(x-2)^{n}}+\frac{1}{(x+2)^{n}}\right]$

15) $\frac{(-1)^{n-1}(n-1)!}{2}\left[\frac{1}{(x+2)^{n}}-\frac{3^{n}}{(3x-1)^{n}}\right]$

16) $\left(\frac{n!}{2a}\right)\left\{(a-x)^{-n-1}+(-1)^{n}(a+x)^{-n-1}\right\}$

17) $(-1)^{n}n!\left[\frac{4}{(x+2)^{n+1}}+\frac{3}{(x+1)^{n+1}}-\frac{n+1}{(x+1)^{n+2}}\right]$

18) $\frac{(-1)^{n}n!}{2}\left[\frac{9\times 2^{n}}{(2x+3)^{n+1}}-\frac{8}{(x+2)^{n+1}}\right]$

19) $(-1)^n n!\left\{\dfrac{1}{(x+1)^{n+1}}-\dfrac{16}{(x+2)^{n+1}}\right\}$

20) $\dfrac{n!}{2}\left[\dfrac{1}{2(1-x)^{n+1}}+\dfrac{(n+1)}{(1-x)^{n+2}}+\dfrac{(-1)^n}{2(1+x)^{n+1}}\right]$

21) $\dfrac{(-1)^n n!}{2}\left[\dfrac{1}{(x+1)^{n+1}}+\dfrac{1}{(x-1)^{n+1}}\right]$

22) $\dfrac{(-1)^n n!}{\left(x^2+a^2\right)^{\frac{(n+1)}{2}}}\cos\left[(n+1)\tan^{-1}\dfrac{a}{x}\right]$

23) $(-1)^n (n!)c^{n-1}(bc-ad)(cx+d)^{-n-1}$

24) $\dfrac{1}{2}\dfrac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta, \quad r=\sqrt{x^2+1},\ \theta=\tan^{-1}\left(\dfrac{1}{x}\right)$

25) $-2\dfrac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta,\ r=\sqrt{x^2+1},\ \theta=\tan^{-1}\left(\dfrac{1}{x}\right)$

26) $\dfrac{(-1)^n(n-1)!}{r^n}\sin n\theta,\ r=\sqrt{x^2+1},\ \theta=\tan^{-1}\left(\dfrac{1}{x}\right)$

27) $\dfrac{(-1)^n(n-1)!}{r^n}\sin n\theta,\ r=\sqrt{x^2+1},\ \theta=\tan^{-1}\left(\dfrac{1}{x}\right)$

28) $(-1)^{n-1}(n-1)!\sin^n\phi\sin n\phi,$ *where* $\phi=\tan^{-1}\left(\dfrac{1}{x}\right)$

LEIBNITZ'S THEOREM

With the help of this theorem, we can find the n^{th} derivative of the product of two functions. If u and v are any two functions of x, then

$$(u.v)_n = y_n = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + --- u_n v$$
$$+ --- {}^nC_r u_{n-r} v_r + --- + u_n v$$

where n is the suffix denotes the n^{th} differentiation.

Worked Examples

1. *Find the n^{th} derivatives of the following*

a) x^2e^x b) $x\cos x$ c) $x^2\log x$

d) x^3e^{2x}

➡ a) Let $y = x^2e^x$

Differentiating n times, we get

$$y_n = D^n\left(x^2e^x\right)$$
$$= x^2D^n\left(e^x\right) + {}^nC_1(2x)D^{n-1}\left(e^x\right) + {}^nC_2(2)D^{n-2}\left(e^x\right)$$
$$= x^2e^x + 2nxe^x + 2\frac{n(n-1)}{2!}e^x = x^2e^x + 2nxe^x + n(n-1)e^x$$

b) Let $y = x\cos x$

Differentiating n times, we get

$$y_n = D^n(x\cos x) = xD^n(\cos x) + {}^nC_1(1)D^{n-1}(\cos x)$$
$$= x\cos\left(x + \frac{n\pi}{2}\right) + n\cos\left(x + \frac{(n-1)\pi}{2}\right)$$

c) Let $y = x^2\log x$

Differentiating n times, we get

$$y_n = D^n\left(x^2\log x\right)$$
$$= x^2D^n(\log x) + {}^nC_1(2x)D^{n-1}(\log x) + {}^nC_2(2)D^{n-3}(\log x)$$
$$= x^2\frac{(-1)^{n-1}(n-1)!}{x^n} + 2nx\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$
$$+ 2\frac{n(n-1)}{2!}\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}$$
$$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + 2n\frac{(-1)^{n-2}(n-2)!}{x^{n-2}}$$
$$+ n(n-1)\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}$$

d) Let $y = x^3e^{2x}$

Differentiating n times, we get

$$y^n = D^n\left(x^3 e^{2x}\right)$$

$$= x^3 D^n\left(e^{2x}\right) + {}^nC_1\left(3x^2\right)D^{n-1}\left(e^{2x}\right) + {}^nC_2(6x)D^{n-2}\left(e^{2x}\right) + {}^nC_3(6)D^{n-3}\left(e^{2x}\right)$$

$$= x^3 2^n e^{2x} + 3nx^2 2^{n-1} e^{2x} + 6x\frac{n(n-1)}{2!}2^{n-2}e^{2x} + 6\frac{n(n-1)(n-2)}{3!}\cdot 2^{n-3}e^{2x}$$

$$= 2^n x^3 e^{2x} + 3\cdot 2^{n-1} nx^2 e^{2x} + 3\cdot 2^{n-2} n(n-1)xe^{2x} + 2^{n-3} n(n-1)(n-2)e^{2x}$$ ■

2. *If* $y = \tan^{-1} x$, *prove that*

$$\left(1+x^2\right)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

➡ Let $y = \tan^{-1} x$

Differentiating with respect to *x*, we get

$$y_1 = \frac{1}{1+x^2}$$

$$\left(1+x^2\right)y_1 = 1$$

Differentiating again with respect to *x*, we get

$$\left(1+x^2\right)y_2 + 2xy_1 = 0$$

Using Leibnitz's theorem, we get

$$\left(1+x^2\right)y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + (2x)y_{n+1} + {}^nC_1(2)y_n = 0$$

$$\left(1+x^2\right)y_{n+2} + n2xy_{n+1} + \frac{n(n-1)}{2!}2y_n + 2xy_{n+1} + n2y_n = 0$$

$$\left(1+x^2\right)y_{n+2} + (n+1)2xy_{n+1} + \left(n^2 - n + 2n\right)y_n = 0$$

$$\left(1+x^2\right)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$ ■

3. *If* $y = \cos(m\log x)$ *then prove that*

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + \left(m^2 + n^2\right)y_n = 0$$

➡ $y = \cos(m\log x)$

Differentiating with respect to *x*, we get

$$y_1 = -\sin(m\log x)\cdot\frac{m}{x}$$

$$xy_1 = -m\sin(m\log x)$$

Differentiating again with respect to x, we get

$$xy_2 + 1\cdot y_1 = -m\cos(m\log x)\cdot\frac{m}{x}$$

$$= -m^2\frac{\cos(m\log x)}{x}$$

$$x^2y_2 + xy_1 = -m^2y$$

$$x^2y_2 + xy_1 + m^2y = 0$$

Differentiating n times, we get

$$x^2y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + xy_{n+1} + {}^nC_1(1)y_n + m^2y_n = 0$$

$$x^2y_{n+2} + 2nxy_{n+1} + 2\frac{n(n-1)}{2!}y_n + xy_{n+1} + ny_n + m^2y_n = 0$$

$$x^2y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n + m^2y_n = 0$$

$$x^2y_{n+2} + 2nxy_{n+1} + n^2y_n - ny_n + xy_{n+1} + ny_n + m^2y_n = 0$$

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (m^2+n^2)y_n = 0$$

4. *If* $y = e^{a\sin^{-1}x}$ *then prove that* $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

➡ $y = e^{a\sin^{-1}x}$

Differentiating with respect to x, we get

$$y_1 = e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = ay$$

Squaring both sides

$$(1-x^2y_1^2) = a^2y^2$$

Differentiating again with respect to x

$$(1-x^2)2y_1y_2 + y_1^2(-2x) = 2yy_1a^2$$

Dividing by $2y_1$ on both the sides, we get

$$\left(1-x^2\right)y_2 - xy_1 = a^2 y$$
$$\left(1-x^2\right)y_2 - xy_1 - a^2 y = 0$$

Differentiating *n* times, we get

$$\left(1-x^2\right)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n$$
$$-\left\{xy_{n+1} + {}^nC_1(1)y_n\right\} - a^2 y_n = 0$$
$$\left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - 2\frac{n(n-1)}{2!}y_n - xy_{n+1} - ny_n - a^2 y_n = 0$$
$$\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left[n^2 - n\right]y_n - ny_n - a^2 y_n = 0$$
$$\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n + ny_n - ny_n - a^2 y_n = 0$$
$$\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2 + a^2\right)y_n = 0$$ ■

5. *If* $y = \sin\left(m\sin^{-1}x\right)$, *then prove that* $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} + \left(m^2 - n^2\right)y_n = 0$

➡ $y = \sin\left(m\sin^{-1}x\right)$

Differentiating with respect to *x*, we get

$$y_1 = \cos\left(m\sin^{-1}x\right)\cdot\frac{m}{\sqrt{1-x^2}}$$
$$\sqrt{1-x^2}\,y_1 = m\cos\left(m\sin^{-1}x\right)$$

Squaring on both sides, we get

$$\left(1-x^2\right)y_1^2 = m^2\cos^2\left(m\sin^{-1}x\right)$$
$$\left(1-x^2\right)y_1^2 = m^2\left(1-\sin^2\left(m\sin^{-1}x\right)\right)$$
$$\left(1-x^2\right)y_1^2 = m^2\left(1-y^2\right)$$

Differentiating with respect to *x*, we get

$$\left(1-x^2\right)\cdot 2y_1y_2 + (-2x)y_1^2 = m^2\left(-2yy_1\right)$$

Dividing by $2y_1$ on both sides, we get

$$\left(1-x^2\right)y_2 - xy_1 = -m^2 y$$
$$\left(1-x^2\right)y_2 - xy_1 + m^2 y = 0$$

Differentiating *n* times, we get

$$\left(1-x^2\right)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n$$
$$-\left\{xy_{n+1} + {}^nC_1(1)y_n\right\} + m^2 y_n = 0$$

$$\left(1-x^2\right)y_{n+2}-2nxy_{n+1}-2\frac{n(n-1)}{2!}y_n-xy_{n+1}-ny_n+m^2y_n=0$$

$$\left(1-x^2\right)y_{n+2}-2nxy_{n+1}-[n(n-1)y_n]-xy_{n+1}$$
$$-ny_n+m^2y_n=0$$

$$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n+ny_n-ny_n+m^2y_n=0$$

$$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}+\left(m^2-n^2\right)y_n=0$$ ■

6. *If* $y=\sin\log\left(x^2+2x+1\right)$, *prove that*

$$(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+\left(n^2+4\right)y_n=0$$

➡ $y=\sin\log\left(x^2+2x+1\right)=\sin\log(x+1)^2$

$$y=\sin[2\log(x+1)]$$

Differentiating with respect to x, we get

$$y_1=\cos[2\log(x+1)]2.\frac{1}{x+1}$$

$$(x+1)y_1=2\cos[2\log(x+1)]$$

Differentiating again with respect to x, we get

$$(x+1)y_2+1y_1=2\left\{-\sin[2\log(x+1)]2\frac{1}{x+1}\right\}$$

$$=-\frac{4\sin[2\log(x+1)]}{x+1}$$

$$(x+1)y_2+y_1=-\frac{4y}{x+1}$$

$$(x+1)^2y_2+(x+1)y_1+4y=0$$

Differentiating n times, we get

$$(x+1)^2y_{n+2}+{}^nC_1[2(x+1)]y_{n+1}+{}^nC_2[2]y_n+(x+1)y_{n+1}$$
$$+{}^nC_1(1)y_n+4y_n=0$$

$$(x+1)^2y_{n+2}+2n(x+1)y_{n+1}+2\frac{n(n-1)}{2!}y_n$$
$$+(x+1)y_{n+1}+ny_n+4y_n=0$$

$$(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+n(n-1)y_n+ny_n+4y_n=0$$

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + [n(n-1)+n+4]y_n = 0$$
$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + [n^2-n+n+4]y_n = 0$$
$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$ ■

7. *If* $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^p$, *prove that*

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+p^2)y_n = 0$$

➡ Given $\cos^{-1}(y/b) = \log(x/n)^p$

$$\frac{y}{b} = \cos\left(\log(x/n)^p\right)$$
$$y = b\cos[p\log(x/n)] = b\cos[p\log x - p\log n]$$

Differentiating with respect to x, we get

$$y_1 = b[-\sin(p\log x - p\log n)\cdot(p(1/x)-0)]$$
$$y_1 = \frac{-bp}{x}\sin(p\log x - p\log n)$$
$$xy_1 = -bp\sin(p\log x - p\log n)$$

Differentiating again with respect to x, we get

$$xy_2 + 1\cdot y_1 = -bp\cos(p\log x - p\log n)\cdot(p(1/x)-0)$$
$$xy_2 + y_1 = -\frac{bp^2}{x}\cos(p\log x - p\log n)$$
$$x^2y_2 + xy_1 = -p^2 y$$
$$x^2y_2 + xy_1 + p^2 y = 0$$

Differentiating n times, we get

$$x^2 y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + xy_{n+1} + {}^nC_1(1)y_n + p^2 y_n = 0$$
$$x^2 y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!}(2)y_n + xy_{n+1} + ny_n + p^2 y_n = 0$$
$$x^2 y_{n+2} + (2n+1)xy_{n+1} + [n(n-1)+n+p^2]\ y_n = 0$$

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + \left[n^2 - n + n + p^2\right]\, y_n = 0$$

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 + p^2)y_n = 0$$ ■

8. *If* $y = \left(x+\sqrt{x^2+1}\right)^m$, *prove that*
$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

➡ $y = \left(x+\sqrt{x^2+1}\right)^m$

Differentiating with respect to x, we get

$$y_1 = m\left(x+\sqrt{x^2+1}\right)^{m-1}\left[1+\frac{1}{2\sqrt{x^2+1}}2x\right]$$

$$y_1 = m\left(x+\sqrt{x^2+1}\right)^{m-1}\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$= \frac{m\left(x+\sqrt{x^2+1}\right)^{m-1+1}}{\sqrt{x^2+1}} = \frac{m\left(x+\sqrt{x^2+1}\right)^{m}}{\sqrt{x^2+1}}$$

$$y_1 = \frac{my}{\sqrt{x^2+1}}$$

$$y_1\sqrt{x^2+1} = my$$

Squaring on both the sides, we get

$$(x^2+1)y_1^2 = m^2y^2$$

Differentiating again, we get

$$(x^2+1)2y_1y_2 + 2xy_1^2 = m^2\,2yy_1$$

Dividing by $2y_1$ on both the sides, we get

$$(x^2+1)y_2 + xy_1 = m^2y$$

$$(x^2+1)y_2 + xy_1 - m^2y = 0$$

Differentiating n times, we get

$$(x^2+1)y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + xy_{n+1} + {}^nC_1(1)y_n - m^2y_n = 0$$

$$(x^2+1)y_{n+2} + 2nxy_{n+1} + 2\frac{n(n-1)}{2!}y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$

$$(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + \left[n(n-1)+n-m^2\right]\ y_n = 0$$

$$(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + \left[n^2-n+n-m^2\right]\ y_n = 0$$

$$(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$ ■

9. *If* $y = a\cos(\log x) + b\sin(\log x)$, *prove that*
$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

➡ $y = a\cos(\log x) + b\sin(\log x)$

Differentiating with respect to *x*, we get

$$y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again, we get

$$xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

$$x^2y_2 + xy_1 = -\left[a\cos(\log x) + b\sin(\log x)\right]$$

$$x^2y_2 + xy_1 = -y$$

$$x^2y_2 + xy_1 + y = 0$$

Differentiating *n* times, we get

$$x^2y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + xy_{n+1} + {}^nC_1(1)y_n + y_n = 0$$

$$x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2\,!}(2)y_n + xy_{n+1} + ny_n + y_n = 0$$

$$x^2y_{n+2}(2n+1)xy_{n+1} + \left[n^2-n+n+1\right]\ y_n = 0$$

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$ ■

10. *If* $y_n = \frac{d^n}{dx^n}\left(x^n \log x\right)$, *prove that* $y_n = (n-1)! + ny_{n-1}$

Deduce that $y_n = n!\left[\log x + 1 + \frac{1}{2} + \ldots\ldots\ldots + \frac{1}{n}\right]$

➡ Given $y_n = \frac{d^n}{dx^n}\left(x^n \log x\right)$

$$y_n = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}\left(x^n \log x\right)\right] = \frac{d^{n-1}}{dx^{n-1}}\left[x^n \frac{1}{x} + nx^{n-1}\log x\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\right) + n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\log x\right)$$

$$y_n = (n-1)! + ny_{n-1}$$

$$y_n = ny_{n-1} + (n-1)!$$

$$y_1 = y + 0! = y + 1$$

$$y_2 = 2y_1 + 1! = 2[y+1] + 1! \times \frac{2}{2} = 2![y+1] + \frac{2!}{2}$$

$$= 2!\left[y + 1 + \frac{1}{2}\right]$$

$$y_3 = 3y_2 + 2! = 3.2!\left[y + 1 + \frac{1}{2}\right] + 2! \times \frac{3}{3} = 3!\left[y + 1 + \frac{1}{2}\right] + \frac{3!}{3}$$

$$y_3 = 3!\left[y + 1 + \frac{1}{2} + \frac{1}{3}\right]$$

....

$$y_n = n!\left[y + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}\right]$$

$$= n!\left[x^0 \log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}\right]$$

$$y_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}\right]$$ ■