$$= \frac{3+i}{6+2i} = \frac{3+i}{2(3+i)}$$

$$z = \frac{1}{2} = \frac{1}{2} + 0.i = \left(\frac{1}{2}, 0\right) \text{say}$$

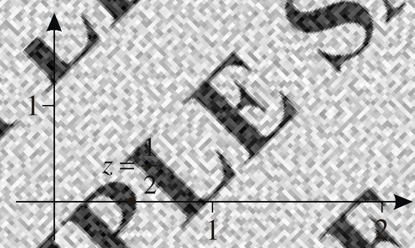

Figure 1.8

(iii) Let $z = \frac{1}{(1-i)^2} - \frac{1}{(1+i)^2}$

$$= \frac{1}{1^2+i^2-2i} - \frac{1}{1^2+i^2+2i}$$

$$= \frac{1}{-2i} - \frac{1}{2i} = -\frac{1}{2i} - \frac{1}{2i} = -\frac{1}{i}$$

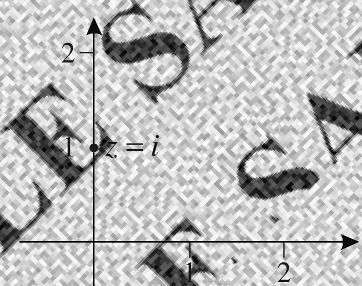

Figure 1.9

$z = i = (0,1)$ say ■

4. *Express the following in the form* $a+ib$ *and also find the conjugate*

(i) $\frac{3}{1+i}-\frac{1}{2-i}+\frac{1}{1-i}$ (ii) $\frac{1}{1-\cos\theta+2i\sin\theta}$ (iii) $\frac{1}{1-i\tan\alpha}$

➡ (i) Let $z=\frac{3}{1+i}-\frac{1}{2-i}+\frac{1}{1-i}=\frac{3}{1+i}+\frac{1}{1-i}-\frac{1}{2-i}$

$$=\frac{3(1-i)+(1+i)}{(1+i)(1-i)}-\frac{1}{2-i}=\frac{3-3i+1+i}{1-i^2}-\frac{1}{2-i}$$

$$=\frac{4-2i}{1+1}-\frac{1}{2-i}$$

$$=2-i-\left(\frac{1}{2-i}\times\frac{2+i}{2+i}\right)=2-i-\left(\frac{2+i}{4-i^2}\right)$$

$$=2-i-\frac{(2+i)}{5}=\frac{10-5i-(2+i)}{5}=\frac{8-6i}{5}$$

$$z=\frac{8}{5}-\frac{6i}{5}$$

Therefore, $\bar{z}=\frac{8}{5}+\frac{6}{5}i$

(ii) Let $z=\frac{1}{1-\cos\theta+i\sin\theta}$

$$=\frac{1}{2\sin^2(\theta/2)+2i\sin(\theta/2)\cos(\theta/2)}$$

$$=\frac{1}{2\sin(\theta/2)(\sin(\theta/2)+i\cos(\theta/2))}$$

$$=\frac{1}{2\sin\left(\frac{\theta}{2}\right)\left(\sin\left(\frac{\theta}{2}\right)+i\cos\left(\frac{\theta}{2}\right)\right)}\times\frac{\left(\sin\left(\frac{\theta}{2}\right)-i\cos\left(\frac{\theta}{2}\right)\right)}{\left(\sin\left(\frac{\theta}{2}\right)-i\cos\left(\frac{\theta}{2}\right)\right)}$$

$$=\frac{\sin(\theta/2)-i\cos(\theta/2)}{2\sin(\theta/2)\left[\sin^2(\theta/2)-i^2\cos^2(\theta/2)\right]}$$

$$= \frac{\sin(\theta/2) - i\cos(\theta/2)}{2\sin(\theta/2)\left(\sin^2(\theta/2) + \cos^2(\theta/2)\right)}$$

$$= \frac{\sin(\theta/2) - i\cos(\theta/2)}{2\sin(\theta/2)(1)}$$

$$= \frac{1}{2}\left(\frac{\sin(\theta/2)}{\sin(\theta/2)} - i\frac{\cos(\theta/2)}{\sin(\theta/2)}\right)$$

$$z = \frac{1}{2}\left(1 - i\cot\left(\frac{\theta}{2}\right)\right)$$

Therefore, $\bar{z} = \frac{1}{2}\left(1 + i\cot\left(\frac{\theta}{2}\right)\right)$

(iii) Let $z = \frac{1}{1 - i\tan\alpha} \times \frac{1 + i\tan\alpha}{1 + i\tan\alpha}$

$$= \frac{1 + i\tan\alpha}{1 - i^2\tan^2\alpha} = \frac{1 + i\tan\alpha}{1 + \tan^2\alpha}$$

$$= \frac{1 + i\tan\alpha}{\sec^2\alpha} = \frac{1}{\sec^2\alpha} + \frac{\tan\alpha}{\sec^2\alpha}$$

$$= \cos^2\alpha + i\cos^2\alpha \tan\alpha$$

$$= \cos^2\alpha + i\cos\alpha\sin\alpha$$

Therefore, $\bar{z} = \cos^2\alpha - i\cos\alpha\sin\alpha$ ■

5. *Find the real values of* θ *if* $\frac{1 - i\sin\theta}{1 + i\sin\theta}$ *is purely real.*

➡ Let $z = \frac{1 - i\sin\theta}{1 + i\sin\theta} \times \frac{1 - i\sin\theta}{1 - i\sin\theta}$

$$= \frac{(1 - i\sin\theta)^2}{1 - i^2\sin^2\theta}$$

$$= \frac{1 + i^2\sin^2\theta - 2i\sin\theta}{1 + \sin^2\theta}$$

$$= \frac{1-\sin^2\theta - 2i\sin\theta}{1+\sin^2\theta}$$

$$= \frac{\cos^2\theta}{1+\sin^2\theta} - i\frac{2\sin\theta}{1+\sin^2\theta}$$

Since z is purely real $\therefore \operatorname{Im}(z) = 0$

i.e., $-\dfrac{2\sin\theta}{1+\sin^2\theta} = 0$

$\sin\theta = 0$

$\Rightarrow \theta = n\pi$ where $n = 0, 1, 2, \ldots..$ ■

6. *Find z if* $(3x-2iy)(2+i)^2 = 10(1+i)$ *where* $z = x+iy$

➡ $(3x-2iy)(2+i)^2 = 10(1+i)$

$(3x-2iy)(2^2+i^2+2.2.i) = 10(1+i)$

$(3x-2iy)(4-1+4i) = 10(1+i)$

$(3x-2iy)(3+4i) = 10(1+i)$

$9x-6iy+12ix-8i^2y = 10+10i$

$(9x+8y)+i(12x-6y) = 10+10i$

Equating real and imaginary parts, we get

$9x+8y = 10$ ---(1)

And $12x-6y = 10$ ---(2)

Multiplying (1) by 3 and (2) by 4 and adding, we get

$75x = 70$

$x = \dfrac{14}{15}$ $\therefore y = \dfrac{1}{5}$

Therefore, $z = x+iy = \dfrac{14}{15} + \dfrac{i}{5}$

$z = \dfrac{14+3i}{15}$ ■

7. *Find the modulus and amplitude of each of the following and express each in polar form.*

(i) $1-i$ (ii) $\sqrt{3}+i$ (iii) $1-i\sqrt{3}$

(iv) $-\sqrt{3}-i$ (v) $2i$ (vi) -8

➡ (i) Let $z = x+iy = 1-i$

$\therefore x = 1, y = -1$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{1}\right| = \tan^{-1}1 = \frac{\pi}{4}$$

Here $z = (x, y) = (1, -1)$ is in the 4^{th} quadrant

Therefore, amp $z = 2\pi - \alpha = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$$\Rightarrow \theta = \frac{7\pi}{4}$$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2} = \text{modulus}$$

The polar form of the given complex number is

$$1-i = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow 1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

(ii) Let $z = x+iy = \sqrt{3}+i$

$$\Rightarrow x = \sqrt{3}, y = 1$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Here $z = (x, y) = (\sqrt{3}, 1)$ is in the first quadrant

Therefore, amp $z = \alpha = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

The polar form of the given complex number is

$$\sqrt{3}+i = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow \sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

(iii) Let $z=x+iy=1-i\sqrt{3}$

$$\Rightarrow x=1, y=-\sqrt{3}$$

$$\alpha=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{-\sqrt{3}}{1}\right|=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$$

Here, $z=(x,y)=(1,-\sqrt{3})$ is in fourth quadrant

Therefore, $\text{amp}\, z=2\pi-\alpha=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}\Rightarrow\theta=5\pi/3$

$$|z|=r=\sqrt{x^2+y^2}=\sqrt{1^2+(-\sqrt{3})^2}=\sqrt{1+3}=2$$

The polar form of the given complex is

$$1-i\sqrt{3}=r(\cos\theta+i\sin\theta)$$

$$\Rightarrow 1-i\sqrt{3}=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$$

(iv) Let $z=x+iy=-\sqrt{3}-i$

$$\Rightarrow x=-\sqrt{3}, y=-1$$

$$\alpha=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{-1}{-\sqrt{3}}\right|=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

Here $z=(x,y)=(-\sqrt{3},-1)$ is in the 3rd quadrant.

Therefore, $\text{amp}\, z=\alpha+\pi=\frac{\pi}{6}+\pi=\frac{7\pi}{6}\Rightarrow\theta=\frac{7\pi}{6}$

$$|z|=r=\sqrt{x^2+y^2}=\sqrt{3+1}=2$$

The polar form of the given complex number is

$$-\sqrt{3}-i=r(\cos\theta+i\sin\theta)$$

$$\Rightarrow -\sqrt{3}-i=2\left[\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right]$$

(v) Let $z = x + iy = 2i$

$\Rightarrow x = 0 \quad y = 2$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{2}{0}\right| = \tan^{-1}|\infty| = \frac{\pi}{2}$$

Here $z = (x, y) = (0,2)$ is in the 1st quadrant

Therefore, $\text{amp}\, z = \theta = \alpha = \frac{\pi}{2}$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{0+4} = 2$$

Therefore, the polar form of given complex number is

$$2i = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow 2i = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

(vi) Let $z = x + iy = -8$

$\Rightarrow x = -8 \quad y = 0$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{0}{-8}\right| = \tan^{-1}(0) = 0$$

Here $z = (x, y) = (-8,0)$ is in the 2nd quadrant

Therefore, $\text{amp}\, z = \theta = \pi - \alpha = \pi - 0 = \pi$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{64+0} = 8$$

Therefore, polar form of the given complex number

$$-8 = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow -8 = 8[\cos\pi + i\sin\pi]$$ ■

8. *Find the modulus and amplitude of the following complex numbers*

(i) $\frac{(3-\sqrt{2}i)^2}{1+2i}$ (ii) $\frac{(1+i)^2}{3+i}$ (iii) $\frac{(2-3i)(2+i)^2}{1+i}$

➡ Let $z = \frac{(3-\sqrt{2}i)^2}{1+2i}$

$$= \frac{9+2i^2-6\sqrt{2}i}{1+2i} = \frac{9-2-6\sqrt{2}i}{1+2i} \times \frac{1-2i}{1-2i}$$

$$= \frac{(7-6\sqrt{2}i)(1-2i)}{1-2^2i^2} = \frac{7-14i-6\sqrt{2}i+12\sqrt{2}i^2}{1+4}$$

$$= \frac{7-i(14+6\sqrt{2})-12\sqrt{2}}{5} = \frac{(7-12\sqrt{2})-i(14+6\sqrt{2})}{5}$$

$$z = \left(\frac{7-12\sqrt{2}}{5}\right) - i\left(\frac{14+6\sqrt{2}}{5}\right)$$

Therefore,

$$|z| = \sqrt{\left(\frac{7-12\sqrt{2}}{5}\right)^2 + \left(\frac{14+6\sqrt{2}}{5}\right)^2}$$

$$= \sqrt{\frac{49+288-2\times7\times12\sqrt{2}}{25} + \frac{196+72+168\sqrt{2}}{25}}$$

$$= \sqrt{\frac{337-168\sqrt{2}+268+168\sqrt{2}}{25}}$$

$$= \sqrt{\frac{605}{25}} = \sqrt{\frac{121}{5}} = \frac{11}{\sqrt{5}}$$

$$\text{amp}z = \tan^{-1}\left(\frac{-\left(\frac{14+6\sqrt{2}}{5}\right)}{\left(\frac{7-12\sqrt{2}}{5}\right)}\right)$$

$$= \tan^{-1}\left(\frac{14+6\sqrt{2}}{12\sqrt{2}-7}\right)$$

(ii) Let $z = \frac{(1+i)^2}{3+i} = \frac{1+i^2+2i}{3+i} = \frac{1-1+2i}{3+i}$

$$= \frac{2i}{3+i} \times \frac{3-i}{3-i} = \frac{6i-2i^2}{9-i^2} = \frac{6i+2}{10}$$

$$z = \frac{1}{5} + \frac{3}{5}i$$

Therefore,

$$|z| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{1}{25} + \frac{9}{25}} = \sqrt{\frac{10}{25}} = \sqrt{\frac{2}{5}}$$

and $\quad \text{amp}z = \tan^{-1}\left(\frac{3/5}{1/5}\right) = \tan^{-1}(3)$

(iii) Let $z = \frac{(2-3i)(2+i)^2}{1+i}$

$$= \frac{(2-3i)(4+i^2+4i)}{1+i} = \frac{(2-3i)(4-1+4i)}{4i}$$

$$= \frac{(2-3i)(3+4i)}{1+i} = \frac{6+8i-9i-12i^2}{1+i}$$

$$= \frac{18-i}{1+i} \times \frac{1-i}{1-i} = \frac{18-18i-i+i^2}{1^2-i^2} = \frac{17-19i}{2}$$

$$z = \frac{17}{2} - \frac{19}{2}i$$

Therefore, $\quad |z| = \sqrt{\left(\frac{17}{2}\right)^2 + \left(-\frac{19}{2}\right)^2} = \sqrt{\frac{325}{2}}$

and $\quad \text{amp}z = \tan^{-1}\left(\frac{-19/2}{17/2}\right) = \tan^{-1}\left(-\frac{19}{17}\right)$ ■

9. *Express the following in polar form and hence find their modulus and amplitude.*

(i) $\sqrt{3} + i$ (ii) $1 + i$ (iii) $1 - i\sqrt{3}$ (iv) i

➡ (i) Let $z = \sqrt{3} + i = 2\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$

$z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$ is in polar form.

Where, modulus $= r = 2$ and amplitude $= \theta = \frac{\pi}{6}$

(ii) Let $z = 1 + i = \sqrt{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$

$$z = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \text{ is in polar form.}$$

Where, modulus $= r = \sqrt{2}$ and amplitude $= \theta = \frac{\pi}{4}$

(iii) Let $z = 1 - \sqrt{3}i = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$

$$= 2\left[\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right]$$

$$= 2\left[\cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)\right]$$

$$z = 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] \text{ is in polar form.}$$

Where, modulus $= r = 2$ and amplitude $= \theta = \frac{5\pi}{3}$

(iv) Let $z = i = 0 + i$

$$z = 1\cdot\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) \text{ is in polar form.}$$

Where, modulus $= r = 1$ and amplitude $= \theta = \frac{\pi}{2}$ ∎

10. *Find the modulus and amplitude of each of the following complex numbers.*

(i) $1 - \cos\alpha + i\sin\alpha$ (ii) $1 + \cos\alpha + i\sin\alpha$

(iii) $\frac{1}{1 - \cos\alpha + i\sin\alpha}$ (iv) $\frac{1}{\sin a - i\cos a}$

(v) $\frac{1}{1 + i\cot\alpha}$ (vi) $\frac{1 + \cos\alpha + i\sin\alpha}{1 - \cos\alpha + i\sin\alpha}$

➡ (i) Let $z = 1 - \cos\alpha + i\sin\alpha$

$$= 2\sin^2\left(\frac{\alpha}{2}\right) + i2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)$$

$$= 2\sin\left(\frac{\alpha}{2}\right)\left(\sin\left(\frac{\alpha}{2}\right) + i\cos\left(\frac{\alpha}{2}\right)\right)$$

$$\therefore z = 2\sin\left(\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$

Since cos and sin are positive in the 1st quadrant

Therefore, $|z| = 2\sin\left(\frac{\alpha}{2}\right)$ and amp $z = \frac{\pi}{2} - \frac{\alpha}{2}$

Remark Polar form of a complex number is

$$z = r(\cos\theta + i\sin\theta)$$

$$|z| = r \text{ and amp } z = \theta$$

(ii) Let $z = 1 + \cos\alpha + i\sin\alpha$

$$= 2\cos^2\left(\frac{\alpha}{2}\right) + i2\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)$$

$$z = 2\cos\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)$$

Therefore, $|z| = 2\cos\left(\frac{\alpha}{2}\right)$ and amp $z = \frac{\alpha}{2}$

(iii) Let $z = \dfrac{1}{1 - \cos\alpha + i\sin\alpha}$

$$= \frac{1}{2\sin^2\left(\frac{\alpha}{2}\right) + i2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}$$

$$= \frac{1}{2\sin\left(\frac{\alpha}{2}\right)\left(\sin\left(\frac{\alpha}{2}\right) + i\cos\left(\frac{\alpha}{2}\right)\right)} \times \frac{\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)\left(\sin^2\left(\frac{\alpha}{2}\right) - i^2\cos^2\left(\frac{\alpha}{2}\right)\right)}$$

$$= \frac{\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)\left(\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right)\right)}$$

$$= \frac{\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)(1)}$$

$$= \frac{1}{2}\operatorname{cosec}\left(\frac{\alpha}{2}\right)\left(\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)\right)$$

$$z = \frac{1}{2}\operatorname{cosec}\left(\frac{\alpha}{2}\right)\left[\cos\left(\frac{3\pi}{2} + \frac{\alpha}{2}\right) + i\sin\left(\frac{3\pi}{2} + \frac{\alpha}{2}\right)\right]$$

Since cos is positive and sin is negative in the 4th quadrant

Therefore, $|z| = \frac{1}{2}\operatorname{cosec}\left(\frac{\alpha}{2}\right)$ and $\operatorname{amp} z = \frac{3\pi}{2} + \frac{\alpha}{2}$

(iv) Let $z = \frac{1}{\sin a - i\cos a} \times \frac{\sin a + i\cos a}{\sin a + i\cos a}$

$$= \frac{\sin a + i\cos a}{\sin^2 a - i^2\cos^2 a}$$

$$= \frac{\sin a + i\cos a}{\sin^2 a + \cos^2 a}$$

$$= \sin a + i\cos a$$

$$z = \cos\left(\frac{\pi}{2} - a\right) + i\sin\left(\frac{\pi}{2} - a\right)$$

Since cos and sin are positive in the 1st quadrant

$$|z| = 1 \text{ and } \operatorname{amp} z = \frac{\pi}{2} - a$$

(v) Let $z = \dfrac{1}{1+i\cot\alpha} \times \dfrac{1-i\cot\alpha}{1-i\cot\alpha}$

$$= \frac{1-i\cot\alpha}{1^2 - i^2\cot^2\alpha} = \frac{1-i\cot\alpha}{1+\cot^2\alpha}$$

$$= \frac{1-i\cot\alpha}{\operatorname{cosec}^2\alpha} = \sin^2\alpha\left(1 - i\frac{\cos\alpha}{\sin\alpha}\right)$$

$$= \sin\alpha(\sin\alpha - i\cos\alpha)$$

$$z = \sin\alpha\left[\cos\left(\frac{3\pi}{2}+\alpha\right) + i\sin\left(\frac{3\pi}{2}+\alpha\right)\right]$$

Since cos is positive and sin is negative in the 4th quadrant

$$|z| = \sin\alpha \text{ and } \operatorname{amp} z = \frac{3\pi}{2} + \alpha$$

(vi) Let $z = \dfrac{1+\cos\alpha + i\sin\alpha}{1-\cos\alpha + i\sin\alpha}$

$$= \frac{2\cos^2\left(\frac{\alpha}{2}\right) + i2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right) + i2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}$$

$$= \frac{2\cos\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)}{2\sin\left(\frac{\alpha}{2}\right)\left(\sin\left(\frac{\alpha}{2}\right) + i\cos\left(\frac{\alpha}{2}\right)\right)} \times \frac{\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right) - i\cos\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\cot\frac{\alpha}{2}\left[\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + i\sin^2\frac{\alpha}{2} - i\cos^2\frac{\alpha}{2} - i^2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]}{\sin^2\frac{\alpha}{2} - i^2\cos^2\frac{\alpha}{2}}$$

$$= \frac{\cot\left(\frac{\alpha}{2}\right)\left[2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)+i\left(\sin^2\left(\frac{\alpha}{2}\right)-\cos^2\left(\frac{\alpha}{2}\right)\right)\right]}{\sin^2\left(\frac{\alpha}{2}\right)+\cos^2\left(\frac{\alpha}{2}\right)}$$

$$= \cot\left(\frac{\alpha}{2}\right)\left(\sin\alpha - i\left(\cos^2\left(\frac{\alpha}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\right)\right)$$

$$= \cot\left(\frac{\alpha}{2}\right)(\sin\alpha - i\cos\alpha)$$

$$z = \cot\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{3\pi}{2}+\alpha\right)+i\sin\left(\frac{3\pi}{2}+\alpha\right)\right)$$

Since cos is positive and sin is negative in the 4th quadrant

$$|z| = \cot\left(\frac{\alpha}{2}\right) \text{ and } \operatorname{amp} z = \frac{3\pi}{2}+\alpha$$ ■

11. *Express the following in exponential form*

(i) $-1+i$ (ii) $\frac{1-i}{1+i}$ (iii) $-\frac{\sqrt{3}}{2}-\frac{1}{2}i$

(iv) $\frac{(1+i)(1+2i)}{1+3i}$

➡ (i) Let $z = -1+i$

$$r = \sqrt{(-1)^2+1^2} = \sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}1 = \frac{\pi}{4}$$

Here $z = -1+i = (-1,1)$ is in the 2nd quadrant

Therefore, $\theta = \operatorname{amp} z = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence, exponential form of a given complex number is

$$z = \sqrt{2}\,e^{\frac{3\pi}{4}i}$$