CHAPTER – 3

INTEGRAL CALCULUS

An integral can be considered as an antiderivative. Thus, if the derivative of $F(x)$ is $F'(x) = f(x)$, then an integral of f(x) is F(x).

In practice integrals are necessary for finding solution of differential equations, finding area under the curve, finding solutions using integral transforms etc.

In this chapter we establish a few reduction formulae for integrals, double and triple integrals, special functions like Beta and Gamma functions and simple problems with standard limits, which has connections with physical and engineering problems and leads to great practical importance.

REDUCTION FORMULAE

Reduction Formula for $\int \sin^n x dx$

$$\text{Let } I_n = \int \sin^n xdx,$$

Where n is positive integer.

$$I_n = \int \sin^{n-1} x(\sin x)\, dx.$$

Applying the method of integration by parts, we get

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x(\cos x)(-\cos x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\, dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\left[1 - \sin^2 x\right]dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \left(\sin^{n-2} x - \sin^n x\right)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} xdx - (n-1)\int \sin^n x\, dx$$

$$I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n + (n-1)I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$I_n[1+n-1] = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}I_{n-2}$$

This is the reduction formula for $\int \sin^n x\,dx$.

Evaluation of $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$

We have, $I_n = -\left.\frac{\sin^{n-1} x\cos x}{n}\right|_0^{\frac{\pi}{2}} + \frac{n-1}{n}I_{n-2}$

$$I_n = \frac{n-1}{n}I_{n-2} \qquad \because [\cos\frac{\pi}{2} = 0 \text{ and } \sin 0 = 0]$$

From the above equation, we get

$$I_{n-2} = \frac{n-3}{n-2}I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4}I_{n-6}$$

$$I_{n-6} = \frac{n-7}{n-6}I_{n-8}$$

....

$$I_3 = \frac{2}{3}I_1$$

$$I_2 = \frac{1}{2}I_0$$

$$\therefore I_n = \frac{n-1}{n}I_{n-2}$$

$$= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)} I_{n-4}$$

$$I_n = \frac{(n-1)}{n}\cdot\frac{(n-3)}{(n-2)}\cdot\frac{(n-5)}{(n-4)} I_{n-6}$$

....

$$\therefore I_n = \begin{cases} \dfrac{(n-1)}{n}\cdot\dfrac{(n-3)}{(n-2)}\cdot\dfrac{(n-5)}{(n-4)}---\dfrac{2}{3} I_1 & \text{if } n \text{ is odd} \\ \dfrac{(n-1)}{n}\cdot\dfrac{(n-3)}{(n-2)}\cdot\dfrac{(n-5)}{(n-4)}---\dfrac{1}{2} I_o & \text{if } n \text{ is even} \end{cases}$$

Here $I_1 = \int_0^{\frac{\pi}{2}} \sin^1 x\, dx = -\cos x\Big]_0^{\frac{\pi}{2}} = -(0-1) = 1$

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x\, dx = \int_0^{\frac{\pi}{2}} dx = x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore I_n = \begin{cases} \dfrac{(n-1)}{n}\cdot\dfrac{(n-3)}{(n-2)}\cdot\dfrac{(n-5)}{(n-4)}\cdots\cdots\dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{(n-1)}{n}\cdot\dfrac{(n-3)}{(n-2)}\cdot\dfrac{(n-5)}{(n-4)}\cdots\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

Reduction Formula for $\int \cos^n xdx$

Let $I_n = \int \cos^n xdx$ where n is positive integer.

$$I_n = \int \cos^{n-1} x \cos xdx$$

Applying the method of integration by parts, we get

$$I_n = \cos^{n-1} x(\sin x) - \int (n-1)\cos^{n-2} x(-\sin x)\sin x\, dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\, dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int (\cos^{n-2} x - \cos^n x)dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} xdx - (n-1)\int \cos^n x\, dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n (1 + n - 1) = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}.$$

This is the reduction formula for $\int \cos^n x \, dx$

Evaluation of $\int_0^{\frac{\pi}{2}} \cos^n x dx$

Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$

We have, $I_n = \left. \frac{\cos^{n-1} x \sin x}{n} \right|_0^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$

$$I_n = \frac{n-1}{n} I_{n-2} \qquad \left[\because \cos \frac{\pi}{2} = 0, \ \sin 0 = 0 \right]$$

From the above equation, we have

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$I_{n-6} = \frac{n-7}{n-6} I_{n-8}$$

....

$$I_3 = \frac{2}{3} I_1$$

$$I_2 = \frac{1}{2} I_0$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2} = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} I_{n-4}$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} I_{n-6} \text{ and so on}$$

$$I_n = \begin{cases} \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots\cdots\cdots \frac{2}{3} I_1 & \text{if } n \text{ is odd} \\ \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots\cdots\cdots \frac{1}{2} I_0 & \text{if } n \text{ is even} \end{cases}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \cos^1 x\,dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

and $$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x\,dx = \int_0^{\frac{\pi}{2}} dx = x \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$I_n = \begin{cases} \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots\cdots \frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots\cdots \frac{1}{2} \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

Worked Examples

1. *Evaluate* $\int \sin^6 x\,dx$

➡ Let $I_6 = \int \sin^6 x\,dx$

We know that $I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$

$$\therefore I_6 = -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} I_4$$

$$= -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \left[-\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} I_2 \right]$$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{15}{24}\left[-\frac{1}{2}\sin x\cos x + \frac{1}{2}I_0\right]$$

$$I_6 = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{15}{48}\sin x\cos x + \frac{15}{48}x + c$$ ■

2. *Evaluate* $\int \cos^5 x\, dx$

➡ We know that $I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n} I_{n-2}$

$$\therefore I_5 = \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}I_3$$

$$= \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}\left[\frac{1}{3}\cos^2 x\cdot\sin x + \frac{2}{3}I_1\right]$$

$$= \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\int\cos x\, dx$$

$$\therefore I_5 = \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + c$$ ■

3. *Evaluate* $\int_0^{\frac{\pi}{2}} \sin^8 x dx$

➡ $\int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$ ■

4. *Evaluate* $\int_0^{\frac{\pi}{2}} \cos^6 x dx$

➡ $\int_0^{\frac{\pi}{2}} \cos^6 x dx = \frac{5.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$ ■

5. *Evaluate* $\int_0^{\frac{\pi}{6}} \sin^6 3x dx$

➡ Let $I = \int_0^{\frac{\pi}{6}} \sin^6 3x dx$

Put $3x = t \Rightarrow 3dx = dt \Rightarrow dx = dt/3$

$x = 0 \Rightarrow t = 0,\ x = \pi/6 \Rightarrow t = \pi/2$

$$\therefore I = \int_0^{\frac{\pi}{2}} \sin^6 t \frac{dt}{3} = \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^6 t\, dt$$

$$= \frac{1}{3}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{96}$$ ■

6. *Evaluate* $\int_0^{2\pi} \cos^4 x dx$

➡ Let $I = \int_0^{2\pi} \cos^4 x dx$

$$\left[\therefore \int_0^a f(x)dx = \int_0^{\frac{a}{2}} f(x)dx + \int_0^{\frac{a}{2}} f(a-x)dx\right]$$

$$= \int_0^{\pi} \cos^4 x dx + \int_0^{\pi} [\cos(2\pi - x)]^4 dx$$

$$= \int_0^{\pi} \cos^4 x dx + \int_0^{\pi} \cos^4 x dx \quad = 2\int_0^{\pi} \cos^4 x dx$$

$$= 2\left[\int_0^{\frac{\pi}{2}} \cos^4 x dx + \int_0^{\frac{\pi}{2}} \cos^4(\pi - x) dx\right]$$

$$= 2\left[\int_0^{\frac{\pi}{2}} \cos^4 x dx + \int_0^{\frac{\pi}{2}} \cos^4 x dx\right]$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4 x dx = 4\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi}{4}$$ ■

7. *Evaluate* $\int_0^{\pi} \sin^4 x dx$

➡ Let $I = \int_0^{\pi} \sin^4 x dx$

$$= \int_0^{\frac{\pi}{2}} \sin^4 x dx + \int_0^{\frac{\pi}{2}} [\sin(\pi - x)]^4 dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^4 x dx + \int_0^{\frac{\pi}{2}} \sin^4 x dx = 2\int_0^{\frac{\pi}{2}} \sin^4 x dx = 2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{8}$$ ■

8. *Evaluate* $\int_0^{\pi} x\cos^6 x\,dx$

➡ Let $I = \int_0^{\pi} x\cos^6 x\,dx$

$$\left[\int_0^{\pi} f(x)dx = \int_0^{\pi} f(a-x)dx\right]$$

$$= \int_0^{\pi} (\pi - x)[\cos(\pi - x)]^6 dx$$

$$= \int_0^{\pi} (\pi - x)(\cos^6 x)dx$$

$$= \int_0^{\pi} \pi\cos^6 x dx - \int_0^{\pi} x\cos^6 x\,dx$$

$$I = \pi\int_0^{\pi} \cos^6 x\,dx - I$$

$$\therefore 2I = \pi\int_0^{\pi} \cos^6 x\,dx$$

$$= \pi\left[\int_0^{\frac{\pi}{2}} \cos^6 x\,dx + \int_0^{\frac{\pi}{2}} [\cos(\pi - x)]^6 dx\right]$$

$$= \pi\left[\int_0^{\frac{\pi}{2}} \cos^6 x\,dx + \int_0^{\frac{\pi}{2}} \cos^6 x\,dx\right]$$

$$2I = 2\pi\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$$

$$I = \pi\int_0^{\frac{\pi}{2}} \cos^6 x\,dx = \pi\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{15\pi^2}{96} = \frac{5}{32}\pi^2$$ ■

9. *Evaluate* $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} d\theta$

➡ Let $I = \int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} d\theta$

$$= \int_0^{\pi} \frac{(2\sin(\theta/2)\cos(\theta/2))^4}{(2\cos^2(\theta/2))^2} d\theta \qquad \left[\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]$$

$$= \int_0^{\pi} \frac{2^4 \sin^4(\theta/2)\cos^4(\theta/2)}{2^2\cos^4(\theta/2)} d\theta \qquad \left[1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$$I = 2^2\int_0^{\pi} \sin^4(\theta/2)\, d\theta$$

Put $\frac{\theta}{2} = x \Rightarrow \frac{d\theta}{2} = dx \Rightarrow d\theta = 2dx$

If $\theta = 0 \Rightarrow x = 0$; If $\theta = \pi \Rightarrow x = \pi/2$

$$\therefore I = 4\int_0^{\frac{\pi}{2}} \sin^4 x\, 2dx = 8\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}$$ ■

10. *Evaluate* $\int_0^{\pi} \frac{\sqrt{1-\cos\theta}}{1+\cos\theta}\sin^2\theta\, d\theta$

➡ Let $I = \int_0^{\pi} \frac{\sqrt{1-\cos\theta}}{1+\cos\theta}\sin^2\theta\, d\theta$

$$= \int_0^{\pi} \frac{\sqrt{2\sin^2(\theta/2)}}{2\cos^2(\theta/2)}(2\sin(\theta/2)\cos(\theta/2))^2 d\theta$$

$$= \int_0^{\pi} \frac{\sqrt{2}\sin(\theta/2).4\sin^2(\theta/2)\cos^2(\theta/2)}{2\cos^2(\theta/2)} d\theta$$

$$I = 2\sqrt{2}\int_0^{\pi} \sin^3(\theta/2)\, d\theta$$

Put $(\theta/2) = x \Rightarrow (d\theta/2) = dx \Rightarrow d\theta = 2dx$

If $\theta = 0 \Rightarrow x = 0$, If $\theta = \pi \Rightarrow x = \pi/2$

$$I = 2\sqrt{2}\int_0^{\frac{\pi}{2}} \sin^3 x\ \ 2dx$$

$$= 4\sqrt{2}\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = 4\sqrt{2}\cdot\frac{2}{3} = \frac{8\sqrt{2}}{3}$$ ■

11. *Evaluate* $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx$

➡ Let $I=\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

If $x=0 \Rightarrow \theta = 0$, If $x = a \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{a^4\sin^4\theta}{\sqrt{a^2-a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= a^5\int_0^{\frac{\pi}{2}} \frac{\sin^4\theta\cos\theta}{\sqrt{a^2\left(1-\sin^2\theta\right)}}d\theta$$

$$= a^5\int_0^{\frac{\pi}{2}} \frac{\sin^4\theta\cos\theta}{a\cos\theta}d\theta$$

$$= a^4\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = a^4\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{3a^4\pi}{16}$$ ■

12. *Evaluate* $\int_0^a x^4\sqrt{a^2-x^2}\,dx$

➡ Let $I = \int_0^a x^4\sqrt{a^2-x^2}\,dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

If $x=0 \Rightarrow \theta = 0$, If $x = a \Rightarrow \theta = \pi/2$

$$I = \int_0^{\frac{\pi}{2}} a^4\sin^4\theta\sqrt{a^2-a^2\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= a^6\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta d\theta$$

$$= a^6\int_0^{\frac{\pi}{2}} \sin^4\theta\left(1-\sin^2\theta\right)d\theta$$

$$= a^6\int_0^{\frac{\pi}{2}} \left(\sin^4\theta - \sin^6\theta\right)d\theta$$

$$= a^6 \int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta - a^6 \int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta$$

$$= a^6 \left[\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] - a^6 \left[\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= a^6 \left[\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] \left(1 - \frac{5}{6} \right) = \frac{3a^6 \pi}{16} \left(\frac{1}{6} \right) = \frac{a^6 \pi}{32}$$ ■

13. *Evaluate* $\int_0^{\infty} \frac{x^2}{(1+x^6)^{7/2}} dx$

➡ Let $I = \int_0^{\infty} \frac{x^2}{(1+x^6)^{7/2}} dx$

Put $x^3 = \tan \theta$

$3x^2 dx = \sec^2 \theta \, d\theta \Rightarrow x^2 dx = (1/3) \sec^2 \theta \, d\theta$

If $x = 0 \Rightarrow \theta = 0$, If $x = \infty \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\infty} \frac{x^2 dx}{\left[1 + (x^3)^2 \right]^{7/2}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{3} \sec^2 \theta \, d\theta}{(1 + \tan^2 \theta)^{7/2}} = \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta}{(\sec^2 \theta)^{7/2}} d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta}{\sec^7 \theta} d\theta = \frac{1}{3} \int_0^{\frac{\pi}{2}} \cos^5 \theta \, d\theta$$

$$= \frac{1}{3} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{45}$$ ■

14. *Evaluate* $\int_0^{a} \frac{x^n}{\sqrt{ax - x^2}} dx$, *where n is positive integer*

➡ Let $I = \int_0^{a} \frac{x^n}{\sqrt{ax - x^2}} dx$

Put $x = a \sin^2 \theta$

$$dx = 2a\sin\theta\cos\theta\, d\theta$$

If $x = 0 \Rightarrow \theta = 0$, If $x = a \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{(a\sin^2\theta)^n}{\sqrt{a^2\sin^2\theta - a^2\sin^4\theta}} 2a\sin\theta\cos\theta\, d\theta$$

$$= 2a^n \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\theta\, a\sin\theta\cos\theta . d\theta}{a\sin\theta\sqrt{1-\sin^2\theta}} = 2a^n \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\, d\theta$$

$$= 2a^n \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \qquad [\because 2n \text{ is an even }]$$

$$= \frac{1.3.5\ldots\ldots..(2n-1)}{2.4.6\ldots\ldots..2n}\pi a^n$$ ■

15. *Evaluate* $\int_0^{\infty} \frac{dx}{(x^2+a^2)^n}$, *where n is positive integer greater than* 1

➡ Let $I = \int_0^{\infty} \frac{dx}{(x^2+a^2)^n}$

Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

If $x = 0 \Rightarrow \theta = 0$, If $x = \infty \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{a\sec^2\theta\, d\theta}{[a^2\tan^2\theta + a^2]^n}$$

$$= \int_0^{\frac{\pi}{2}} \frac{a\sec^2\theta\, d\theta}{(a^2\sec^2\theta)^n} = \int_0^{\frac{\pi}{2}} \frac{a\sec^2\theta\, d\theta}{a^{2n}\sec^{2n}\theta}$$

$$= \frac{1}{a^{2n-1}}\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\theta}\cos^{2n}\theta\, d\theta = \frac{1}{a^{2n-1}}\int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta\, d\theta$$

$$= \frac{1}{a^{2n-1}}\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdots\frac{\pi}{2}$$

$$= \frac{1}{a^{2n-1}}\frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)}\cdot\frac{\pi}{2}$$ ■

16. *Evaluate* $\int_0^1 \frac{x^9}{\sqrt{1-x^2}}\,dx$

➡ Let $I = \int_0^1 \frac{x^9}{\sqrt{1-x^2}}\,dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

If $x = 0 \Rightarrow \theta = 0$, If $x = 1 \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sin^9\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^9\theta}{\sqrt{\cos^2\theta}}\cos\theta\cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^9\theta\, d\theta = \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{128}{315}$$

Exercises

Evaluate the following integrals

1) $\int \cos^6 x dx$
2) $\int_0^{\frac{\pi}{2}} \cos^6 x dx$
3) $\int_0^{\frac{\pi}{2}} \sin^4 x dx$
4) $\int_0^{\frac{\pi}{2}} \sin^8 3x dx$
5) $\int_0^{\pi} \cos^6 x dx$
6) $\int_0^{\frac{\pi}{2}} \cos^7 x dx$
7) $\int_0^{\pi} x\cos^4 x dx$
8) $\int_0^{\pi} x\sin^3 x dx$
9) $\int_0^{\pi} (1-\cos\theta)^3 d\theta$
10) $\int_0^1 \frac{x^7}{\sqrt{1-x^4}}dx$
11) $\int_0^{\infty} \frac{dx}{(1+x^2)^5}$
12) $\int_0^{\infty} \frac{x^2}{(x^6+1)^{9/2}}dx$

Answers

1) $\frac{1}{48}\left[8\cos^5 x\sin x + 10\cos^3 x\sin x + 15\cos x\sin x + 15x\right] + c$

2)	$\frac{5\pi}{32}$	3)	$\frac{3\pi}{16}$	4)	$\frac{35\pi}{768}$	5)	$\frac{15\pi}{48}$
6)	$\frac{16}{35}$	7)	$\frac{3\pi^2}{16}$	8)	$\frac{2\pi}{3}$	9)	$\frac{5\pi}{2}$
10)	$\frac{1}{3}$	11)	$\frac{35\pi}{256}$	12)	$\frac{16}{105}$		

Reduction Formula for $\int \sin^m x \cos^n x \, dx$

Let $I_{m,n} = \int \sin^m x \cos^n x \, dx$

Where m and n are positive integers.

$$I_{m,n} = \int \left(\sin^m x \cos x\right) \cos^{n-1} x dx \qquad \text{---(1)}$$

Consider $\int \sin^m x \cos x dx = \int t^m dt = \frac{t^{m+1}}{m+1} = \frac{\sin^{m+1} x}{m+1}$

Applying integration by parts on RHS of equation (1), we get

$$\therefore I_{m,n} = \cos^{n-1} x\left(\frac{\sin^{m+1} x}{m+1}\right) - \int (n-1)\cos^{n-2} x(-\sin x)\left(\frac{\sin^{m+1} x}{m+1}\right)dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x . \cos^{n-2} x \sin^2 x \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x \left(1 - \cos^2 x\right) dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \left(\sin^m x \cos^{n-2} x - \sin^m x \cos^n x\right) dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x \, dx$$

$$- \frac{n-1}{m+1}\int \sin^m x \cos^n x \, dx$$

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\left(\frac{m+1+n-1}{m+1}\right) I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\frac{m+n}{m+1} I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

This is the reduction formula for $\int \sin^m x \cos^n x \, dx$

Evaluation of $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$

Let $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$

We have, $I_{m,n} = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\frac{\pi}{2}} + \frac{n-1}{m+n} I_{m,n-2}$

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

Replacing n by $n-2, n-4, \ldots$, we get

$$I_{m,n-2} = \frac{n-3}{m+n-2} I_{m,n-4}$$

$$I_{m,n-4} = \frac{n-5}{m+n-4} I_{m,n-6}, \quad \text{and so on}$$

Finally, $I_{m,3} = \frac{2}{m+3} I_{m,1}$, when n is odd

$$I_{m,2} = \frac{1}{m+2} I_{m,0}, \quad \text{when } n \text{ is even}$$

$$\therefore I_{m,n}=\begin{cases}\dfrac{(n-1)}{(m+n)}\dfrac{(n-3)}{(m+n-2)}\dfrac{(n-5)}{(m+n-4)}---\dfrac{2}{(m+3)}I_{m,1} \text{ if } n \text{ is odd}\\ \dfrac{(n-1)}{(m+n)}\dfrac{(n-3)}{(m+n-2)}\dfrac{(n-5)}{(m+n-4)}---\dfrac{1}{(m+2)}I_{m,0} \text{ if } n \text{ is even}\end{cases}$$

$$I_{m,1}=\int_0^{\frac{\pi}{2}}\sin^m x\cos x\,dx=\left.\frac{\sin^{m+1}x}{m+1}\right]_0^{\frac{\pi}{2}}=\frac{1}{m+1}(1-0)=\frac{1}{m+1}$$

$$I_{m,0}=\int_0^{\frac{\pi}{2}}\sin^m x\,dx=\begin{cases}\dfrac{(m-1)}{m}\dfrac{(m-3)}{(m-2)}\dfrac{(m-5)}{(m-4)}---\dfrac{2}{3} & \text{if } m \text{ is odd}\\ \dfrac{(m-1)}{m}\dfrac{(m-3)}{(m-2)}\dfrac{(m-5)}{(m-4)}---\dfrac{1}{2}\dfrac{\pi}{2} & \text{if } m \text{ is even}\end{cases}$$

Case (i) If n is odd and m is odd or even

$$I_{m,n}=\frac{(n-1)}{(m+n)}\frac{(n-3)}{(m+n-2)}\cdot\frac{(n-5)}{(m+n-4)}\cdots\cdots\cdots\frac{2}{(m+3)}\frac{1}{(m+1)}$$

Case (ii) If n is even and m is odd

$$I_{m,n}=\frac{(n-1)}{(m+n)}\cdot\frac{(n-3)}{(m+n-2)}\cdot\frac{(n-5)}{(m+n-4)}\cdots\frac{1}{(m+2)}\left[\frac{(m-1)}{m}\frac{(m-3)}{(m-2)}\cdots\frac{2}{3}\right]$$

$$=\frac{[(n-1)(n-3)\ldots3.1][(m-1)(m-3)\ldots2]}{(m+n)(m+n-2)(m+n-4)---3}$$

Case (iii) If n is even and m is even

$$I_{m,n}=\frac{n-1}{m+n}\cdot\frac{n-2}{m+n-2}\cdot\frac{n-5}{m+n-4}\cdots\cdots\frac{1}{m+2}\cdot\left(\frac{m-1}{m}\frac{m-3}{m-2}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$=\frac{[(n-1)(n-3)\ldots3.1][(m-1)(m-3)\ldots1]}{(m+n)(m+n-2)(m+n-4)---4.2}\cdot\frac{\pi}{2}$$

Worked Examples

1. *Evaluate* $\int_0^{\frac{\pi}{2}}\sin^3 x\cos^7 x\,dx$

➡ $I_{3,7}=\int_0^{\frac{\pi}{2}}\sin^3 x\cos^7 x\,dx$