CHAPTER – 4

DIFFERENTIAL EQUATIONS

A large number of basic laws of nature in engineering mathematics, applied sciences etc., appear mathematically in the form of differential equations. The problem of solving differential equations is a general goal of differential and integral calculus and the theory of differential equations is a very important part of mathematics for understanding physical sciences.

In the study of differential equations, we have two types of problems, namely

(i) Formation of differential equations.

(ii) Solution of differential equations.

In this chapter we discuss the first order, first-degree differential equations in depth and higher order differential equations in brief.

Definition

An equation containing one dependent variable and its derivatives with respect to one or more independent variables is called a differential equation.

There are two types of differential equations namely,

(i) Ordinary differential equations (ODEs)

(ii) Partial differential equations (PDEs)

If the dependent variable depends on one independent variable, the differential equation is called an ordinary differential equation. If it depends on two or more variables, the differential equation is called partial differential equation. Here we discuss only ordinary differential equations.

Examples

(i) $\frac{dy}{dx} + x = 0$

(ii) $\frac{dy}{dx} = a + bx + cy$

(iii) $\frac{d^2y}{dx^2} + y = e^{-x}$

(iv) $x^2\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 + 2y = 0$

(v) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(vi) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

Examples (i) – (iv) are ODEs and (v) and (vi) are PDEs.

Order and Degree of an Ordinary Differential Equation

The order of an ODE is the order of the highest derivative present in that equation. The degree is the power to the highest order derivative appearing in an ODE when the differential coefficients are free from radicals and fractions.

In the above examples (i) and (ii) are of first order and first degree, (iii) and (iv) are of second order and first degree.

Now we consider the following ODEs

(vii) $\left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}} = x + y$

Cubing on both sides, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = (x+y)^3$$

This is an ODE of order 2 and degree 2.

(viii) $y - \frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$

Squaring both sides, we get

$$\left(y - \frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3$$

This is an ODE of order 1 and degree 3.

EQUATIONS OF FIRST ORDER AND OF FIRST DEGREE

In this section we discuss some methods for finding the general solution of an ODE of first order and first degree. These methods depend on the form of the ODE.

Here we discuss the following methods.

1. Equations in which variables are separable
2. Homogeneous equations
3. Reducible to homogeneous equations
4. Linear equations
5. Bernoulli's equations (Reducible to linear equations)
6. Exact equations
7. Reducible to exact equations.

Solution of Differential Equation

General Solution

Given an n^{th} order linear ordinary differential equation, the general solution contains all n linearly independent solutions, with a constant multiplying each one. For example, the differential equation $y'' + y = 1$ has the general solution.

$y(x) = 1 + c_1 \sin x + c_2 \cos x$, where c_1 and c_2 are arbitrary constants.

Particular Solution

Given a linear differential equation, $F(y, y', y'', ---) = f(x)$, the general solution can be written as $y(x) = y_p(x) + \sum c_i y_i(x)$, where

$y_p(x)$, the particular solution, is any solution that satisfies $F(y, y', y'', ---) = f(x)$. For example, the differential equation $y'' + y = 0$ has the general solution

$$y(x) = c_1 \sin x + c_2 \cos x$$

If we impose the boundary conditions $y(0) = 1$, $y(\pi/2) = 1$, then $c_1 = c_2 = 1$.

Hence, $y(x) = \sin x + \cos x$ is the particular solution. *That is in the general solution if we know the values for arbitrary constants, then it becomes the particular solution for the given differential equation.*

Singular Solutions

A singular solution of a differential equation that is not derivable from the general solution by any choice of the arbitrary constants appearing in the general solution. Only nonlinear equations have singular solutions

Trivial Solution

The trivial solution is the identically zero solution. For example, in the differential equation $y'' + y = 2$, $y(x) = 2$ is the trivial solution.

SEPARATION OF VARIABLES

If the differential equation can be put in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}, g(y) \neq 0,$$

Then the equation is said to be of variable separable form.

In this case the solution of the given differential equation can be obtained as follows.

$$g(y)dy = f(x)dx$$

On integration, we get

$$\int g(y)dy = \int f(x)dx + c$$

where c is constant of integration.

Worked Examples

1. *Solve* $\frac{dy}{dx} = \frac{1+\cos 2y}{1+\cos 2x}$

➡ $$\frac{dy}{dx} = \frac{1+\cos 2y}{1+\cos 2x}$$

$$\frac{dy}{1+\cos 2y} = \frac{dx}{1+\cos 2x}$$

$$\int \frac{dy}{1+\cos 2y} = \int \frac{dx}{1+\cos 2x} + c$$

$$\int \frac{dy}{2\cos^2 y} = \int \frac{dx}{2\cos^2 x} + c \qquad [\because 1+\cos 2A = 2\cos^2 A]$$

$$\frac{1}{2}\int \sec^2 y dy = \frac{1}{2}\int \sec^2 x dx + c$$

$$\frac{1}{2}\tan y = \frac{1}{2}\tan x + c$$

$\tan y = \tan x + c_1$, where $c_1 = 2c$ ■

2. *Solve* $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

➡ $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$$\sec^2 x \tan y dx = -\sec^2 y \tan x dy$$

$$\frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy + c$$

Put $\tan x = t$ and $\tan y = z$

$$\sec^2 x dx = dt \qquad \sec^2 y dy = dz$$

$$\int \frac{dt}{t} = -\int \frac{dz}{z} + c$$

$\log t = -\log z + \log c_1$, where $c = \log c_1$

$$\log t + \log z = \log c_1$$

$$\log(tz) = \log c_1$$

$$tz = c_1$$

$\tan x \tan y = c_1$ ■

3. *Solve* $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

➡ $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y} = e^x e^{-y} + x^2 e^{-y}$

$$\frac{dy}{dx} = \left(e^x + x^2\right)e^{-y}$$

$$\frac{dy}{e^{-y}} = \left(e^x + x^2\right)dx$$

$$e^y dy = \left(e^x + x^2\right)dx$$

$$\int e^y dy = \int\left(e^x + x^2\right)dx + c$$

$$e^y = e^x + \frac{x^3}{3} + c$$ ■

4. *Solve* $\sec x \ \log(\sec y + \tan y)dx = \sec y \ \log(\sec x + \tan x)dy$

➡ $\sec x \ \log(\sec y + \tan y)dx = \sec y \ \log(\sec x + \tan x)dy$

$$\frac{\sec x dx}{\log(\sec x + \tan x)} = \frac{\sec y dy}{\log(\sec y + \tan y)}$$

$$\int\frac{\sec x dx}{\log(\sec x + \tan x)} = \int\frac{\sec y dy}{\log(\sec y + \tan y)} + c$$

Put $\log(\sec x + \tan x) = t$

$$\frac{1}{\sec x + \tan x}\left(\sec x.\tan x + \sec^2 x\right)dx = dt$$

$$\frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}dx = dt$$

$$\sec x dx = dt$$

Similarly, if $\log(\sec y + \tan y) = z$, then $\sec y dy = dz$

$$\int\frac{dt}{t} = \int\frac{dz}{z} + c$$

$$\log t = \log z + c$$

$$\log t = \log z + \log c_1, \text{ where } c = \log c_1$$

$$\log t = \log(c_1 z)$$

$$t = c_1 z$$

$$\therefore \log(\sec x + \tan x) = c_1 \log(\sec y + \tan y)$$ ■

5. *Solve* $x\sqrt{1+y^2} + y\sqrt{1+x^2}\dfrac{dy}{dx} = 0$

➡ $x\sqrt{1+y^2} + y\sqrt{1+x^2}\dfrac{dy}{dx} = 0$

$$x\sqrt{1+y^2} = -y\sqrt{1+x^2}\frac{dy}{dx}$$

$$\frac{xdx}{\sqrt{1+x^2}} = -\frac{ydy}{\sqrt{1+y^2}}$$

$$\int\frac{xdx}{\sqrt{1+x^2}} + \int\frac{ydy}{\sqrt{1+y^2}} = c$$

Put $1+x^2 = t$ and $1+y^2 = z$

$2xdx = dt \qquad 2ydy = dz$

$xdx = dt/2 \qquad ydy = dz/2$

$$\int\frac{1}{\sqrt{t}}\frac{dt}{2} + \int\frac{1}{\sqrt{z}}\frac{dz}{2} = c$$

$$\frac{1}{2}\int t^{-\frac{1}{2}}dt + \frac{1}{2}\int z^{-\frac{1}{2}}dz = c$$

$$\frac{1}{2}\frac{t^{\frac{1}{2}}}{(1/2)} + \frac{1}{2}\frac{z^{\frac{1}{2}}}{(1/2)} = c$$

$$\sqrt{t} + \sqrt{z} = c$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$ ■

6. *Solve* $xy\dfrac{dy}{dx} = (1+x+y+xy)$

➡ $xy\dfrac{dy}{dx} = (1+x+y+xy) = 1+x+y(1+x)$

$$xy\frac{dy}{dx} = (1+x)(1+y)$$

$$\frac{ydy}{1+y} = \frac{1+x}{x}dx$$

$$\int \frac{y\,dy}{1+y} = \int \frac{x+1}{x}\,dx + c$$

$$\int \frac{y+1-1}{1+y}\,dy = \int \left(1+\frac{1}{x}\right)dx + c$$

$$\int \left(1-\frac{1}{1+y}\right)dy = x + \log x + c$$

$$y - \log(1+y) = x + \log x + c$$

$$y - x = \log(1+y) + \log x + c$$

$$y - x = \log[x(1+y)] + c$$ ■

7. *Solve* $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$

➡ $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$

$$(\sin y + y\cos y)\,dy = x(2\log x+1)\,dx$$

$$-\cos y + y\sin y - \int 1.\sin y\,dy = \int (2\log x+1)\,x\,dx + c$$

$$-\cos y + y\sin y - (-\cos y) = (2\log x+1)\frac{x^2}{2} - \int\left(2\cdot\frac{1}{x}\right)\cdot\frac{x^2}{2}\,dx + c$$

$$y\sin y = \frac{x^2}{2}(2\log x+1) - \frac{x^2}{2} + c$$

$$y\sin y = x^2\log x + \frac{x^2}{2} - \frac{x^2}{2} + c$$

$$y\sin y = x^2\log x + c$$ ■

8. *Solve* $3e^x \tan y\,dx = (1-e^x)\sec^2 y\,dy$

➡ $3e^x \tan y\,dx = (1-e^x)\sec^2 y\,dy$

$$\frac{3e^x}{1-e^x}\,dx = \frac{\sec^2 y}{\tan y}\,dy$$

$$\int \frac{3e^x}{1-e^x}\,dx = \int \frac{\sec^2 y}{\tan y}\,dy + c$$

Put $1-e^x = t$ and $\tan y = z$

$$-e^x dx = dt \quad \sec^2 y\,dy = dz$$

$$e^x dx = -dt$$

$$\therefore 3\int \frac{-dt}{t} = \int \frac{dz}{z} + c$$

$$-3\log t = \log z + c$$

$$3\log t + \log z = -c$$

$$\log t^3 + \log z = \log c_1 \text{ where } \log c_1 = -c$$

$$\log t^3 z = \log c_1$$

$$t^3 z = c_1$$

$$\therefore (1-e^x)^3 \tan y = c_1$$

9. *Solve* $(1+x)^2 \dfrac{dy}{dx} + 1 = 2e^{-y}$

➨ $(1+x)^2 \dfrac{dy}{dx} + 1 = 2e^{-y}$

$$(1+x)^2 \frac{dy}{dx} = 2e^{-y} - 1$$

$$\frac{dy}{2e^{-y}-1} = \frac{dx}{(1+x)^2}$$

$$\int \frac{dy}{2e^{-y}-1} = \int \frac{dx}{(1+x)^2} + c$$

$$\int \frac{dy}{2.(1/e^y)-1} = \int (1+x)^{-2} dx + c$$

$$\int \frac{e^y}{2-e^y}\,dy = \frac{(1+x)^{-1}}{-1} + c$$

Put $2-e^y = t \Rightarrow -e^y dy = dt \Rightarrow e^y dy = -dt$

$$\therefore \int \frac{-dt}{t} = -\frac{1}{1+x} + c$$

$$-\log t = -\frac{1}{1+x} + c$$

$$\log t = \frac{1}{1+x} + c_1, \text{ where } c_1 = -c$$

$$\log\left(2 - e^{y}\right) = \frac{1}{1+x} + c_1$$

10. *Solve* $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$, *given that* $y = 2$ *at* $x = 1$

➡ $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$

$$y - y^2 = \frac{dy}{dx} + x\frac{dy}{dx}$$

$$y(1-y) = (1+x)\frac{dy}{dx}$$

$$\frac{dx}{1+x} = \frac{dy}{y(1-y)}$$

$$\int \frac{dx}{1+x} = \int \frac{dy}{y(1-y)} + c \qquad \text{---(1)}$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} \qquad \text{[by partial fractions]}$$

$$1 = A(1-y) + By$$

Put $y = 0 \Rightarrow 1 = A$, and $y = 1 \Rightarrow B = 1$

$$\text{(1) becomes, } \int \frac{dx}{1+x} = \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy + c$$

$$\log(1+x) = \log y + \frac{\log(1-y)}{(-1)} + \log c_1 \qquad [c = \log c_1]$$

$$\log(1+x) = \log y - \log(1-y) + \log c_1$$