

CHAPTER – 2

DIFFERENTIAL CALCULUS

The students are already familiar with the notions of limits, continuity and the process of differentiating functions of a single independent variable.

In this chapter we discuss Successive Differentiation, Polar Curves, Taylor's Series, Maclaurin's Series and Partial Differentiation with some solved problems.

SUCCESSIVE DIFFERENTIATION

If $y = f(x)$ be a real valued function of x , its differentiation dy/dx will be in general a function of x which can be differentiated. The differentiation of dy/dx is called the second differentiation of y . Similarly, the differentiation of d^2y/dx^2 is called the third differentiation of y , and so on. The process of doing second and higher order differentiation is called successive differentiation. The successive differentiation of y are denoted by one of the following ways.

- (i) $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \dots, \frac{d^n y}{dx^n}$
- (ii) $f'(x), f''(x), f'''(x), \dots, \dots, f^{(n)}(x)$
- (iii) $y_1, y_2, y_3, \dots, \dots, y_n$
- (iv) $y', y'', y''', \dots, \dots, y^{(n)}$
- (v) $Dy, D^2y, D^3y, \dots, \dots, D^n y$

The value of a differentiation at $x = x_1$ is usually denoted by adding a suffix, i.e., $(y^{(n)})_{x=x_1}$ or $(y^{(n)})_{x_1}$

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n^{th} Derivatives of Some Standard Functions**1. n^{th} derivative of $(ax+b)^m$**

$$\text{Let } y = (ax+b)^m$$

$$y_1 = am(ax+b)^{m-1}$$

$$y_2 = a^2 m(m-1)(ax+b)^{m-2}$$

$$y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$$

$$\begin{aligned} y_n &= a^n m(m-1)(m-2)\dots\dots\dots\{m-(n-1)\}(ax+b)^{m-n} \\ &= \frac{a^n (ax+b)^{m-n} m(m-1)(m-2)\dots\dots\dots\{m-(n-1)\}(m-n)\dots\dots\dots 3.2.1}{(m-n)\dots\dots\dots 3.2.1} \end{aligned}$$

$$y_n = \frac{a^n m! (ax+b)^{m-n}}{(m-n)!} \quad \text{---(1)}$$

Special Cases

(i) If $m > n$, then y_n exists and y_n is given by (1)

(ii) If $m = n$, then $y_n = n! a^n$ i.e., $D^n [(ax+b)^n] = n! a^n$

(iii) If $m < n$, then $y_n = 0$

(iv) If $a = 1, b = 0$ and $m = n$, then $y_n = n!$ i.e., $D^n (x^n) = n!$

2. n^{th} derivative of $\frac{1}{ax+b}$

$$\text{Let } y = \frac{1}{ax+b} = (ax+b)^{-1}$$

$$y_1 = (-1)(ax+b)^{-2}(a) = a(-1)(ax+b)^{-2}$$

$$y_2 = a(-1)(-2)(ax+b)^{-3}(a)$$

$$= a^2(-1)(-2)(ax+b)^{-3}$$

$$\dots\dots\dots y_n = a^n (-1)(-2)\dots(-n)(ax+b)^{-(n+1)}$$

$$= a^n (-1)^n [1.2.3\dots n](ax+b)^{-(n+1)}$$

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$$y_n = \frac{a^n (-1)^n n!}{(ax+b)^{n+1}}$$

i.e., $D^n \left(\frac{1}{ax+b} \right) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

3. *nth derivative of e^{ax}*

Let $y = e^{ax}$

$$y_1 = a e^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$y_3 = a^3 e^{ax}$$

.....

$$y_n = a^n e^{ax} \quad \text{i.e., } D^n(e^{ax}) = a^n e^{ax}$$

If $a = 1$, then *nth* derivative of e^x is $y_n = e^x$ i.e., $D^n(e^x) = e^x$

4. *nth derivative of a^x*

Let $y = a^x$

$$y_1 = a^x \log a$$

$$y_2 = a^x (\log a)^2$$

$$y_3 = a^x (\log a)^3$$

.....

$$y_n = a^x (\log a)^n \quad \text{i.e., } D^n(a^x) = (\log a)^n a^x$$

5. *nth derivative of log(ax + b)*

Let $y = \log(ax + b)$

$$y_1 = \frac{a}{(ax+b)} = a(ax+b)^{-1}$$

$$y_2 = (-1)a^2(ax+b)^{-2} = (-1)!a^2(ax+b)^{-2}$$

$$y_3 = (-1)^2 1.2.(ax+b)^{-3} a^3 = (-1)^2 2!a^3(ax+b)^{-3}$$

.....

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$$y_n = (-1)^{n-1} (n-1)! a^n (ax+b)^{-n} = \frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}$$

$$\text{i.e., } D^n [\log(ax+b)] = \frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}$$

If $a=1$ and $b=0$ then $y = \log x$ and $y_n = \frac{(-1)^{n-1} (n-1)!}{x^n}$

6. n^{th} derivative of $\sin(ax+b)$

Let $y = \sin(ax+b)$

$$y_1 = a \cos(ax+b) = a \sin\left(\frac{\pi}{2} + ax+b\right) \quad \because \sin(90+\theta) = \cos\theta$$

$$y_2 = a^2 \cos\left(\frac{\pi}{2} + ax+b\right) = a^2 \sin\left(\frac{2\pi}{2} + ax+b\right)$$

$$y_3 = a^3 \cos(\pi + ax+b) = a^3 \sin\left(\frac{3\pi}{2} + ax+b\right)$$

.....

$$y_n = a^n \sin\left(ax+b + \frac{n\pi}{2}\right)$$

$$\text{i.e., } D^n [\sin(ax+b)] = a^n \sin\left(ax+b + \frac{n\pi}{2}\right)$$

If $a=1$, $b=0$, then $y = \sin x$ and $y_n = \sin\left(x + \frac{n\pi}{2}\right)$

7. n^{th} derivative of $\cos(ax+b)$

Let $y = \cos(ax+b)$

$$y_1 = -a \sin(ax+b) = a \cos\left(\frac{\pi}{2} + ax+b\right) \quad \because \cos(90+\theta) = -\sin\theta$$

$$y_2 = -a^2 \sin\left(\frac{\pi}{2} + ax+b\right) = a^2 \cos\left(\frac{2\pi}{2} + ax+b\right)$$

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$$y_3 = -a^3 \sin(\pi + ax + b) = a^3 \cos\left(\frac{3\pi}{2} + ax + b\right)$$

$$\dots\dots\dots$$

$$y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

$$\text{i.e., } D^n(\cos(ax + b)) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

If $a = 1, b = 0$, then $y = \cos x$ and $y_n = \cos\left(x + \frac{n\pi}{2}\right)$

8. n^{th} derivative of $e^{ax} \sin(bx + c)$

Let $y = e^{ax} \sin(bx + c)$

$$y_1 = e^{ax} b \cos(bx + c) + a e^{ax} \sin(bx + c)$$

$$= e^{ax} \{b \cos(bx + c) + a \sin(bx + c)\}$$

Putting $a = r \cos \theta$ and $b = r \sin \theta$, we have

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{b}{a} \quad \text{where } r \text{ and } \theta \text{ are constants}$$

$$\therefore y_1 = e^{ax} r \{\sin \theta \cos(bx + c) + \cos \theta \sin(bx + c)\}$$

$$= r e^{ax} \sin(bx + c + \theta)$$

Similarly, $y_2 = r \{e^{ax} \cos(bx + c + \theta) b + e^{ax} a \sin(bx + c + \theta)\}$

$$= r e^{ax} \{b \cos(bx + c + \theta) + a \sin(bx + c + \theta)\}$$

$$= r e^{ax} \{r \sin \theta \cos(bx + c + \theta) + r \cos \theta \sin(bx + c + \theta)\}$$

$$= r^2 e^{ax} \sin(bx + c + 2\theta)$$

$$\dots\dots\dots$$

$$y_n = r^n e^{ax} \sin(bx + c + n\theta)$$

$$\therefore y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

$$D^n [e^{ax} \sin(bx + c)] = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + c + n \tan^{-1} (b/a)\right)$$

If $a = 1, b = 1$ and $c = 0$, then

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$$y = e^x \sin x \text{ \& } y_n = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right) \quad \left[\because r = \sqrt{2} \quad \theta = \frac{\pi}{4} \right]$$

9. n^{th} derivative of $e^{ax} \cos(bx + c)$

Let $y = e^{ax} \cos(bx + c)$

$$y_1 = e^{ax} [-b \sin(bx + c)] + ae^{ax} \cos(bx + c)$$

Putting $a = r \cos \theta$ and $b = r \sin \theta$, we get

$$y_1 = re^{ax} \{\cos \theta \cos(bx + c) - \sin \theta \sin(bx + c)\} \\ = re^{ax} \cos(bx + c + \theta)$$

$$y_2 = r \{e^{ax} (-\sin(bx + c + \theta)b) + e^{ax} a \cos(bx + c + \theta)\} \\ = re^{ax} \{a \cos(bx + c + \theta) - b \sin(bx + c + \theta)\} \\ = re^{ax} \{r \cos \theta \cos(bx + c + \theta) - r \sin \theta \sin(bx + c + \theta)\}$$

$$y_2 = r^2 e^{ax} \cos(bx + c + 2\theta)$$

.....

$$\therefore y_n = r^n e^{ax} \cos(bx + c + n\theta)$$

$$\therefore y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

$$\text{i.e., } D^n [e^{ax} \cos(bx + c)] = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + c + n \tan^{-1} (b/a)\right)$$

If $a = 1$, $b = 1$, $c = 0$, then

$$y = e^x \cos x \text{ and } y_n = 2^{\frac{n}{2}} e^x \cos\left(x + n(\pi/4)\right)$$

The formulae list for n^{th} derivatives

$$D^n (e^{ax}) = a^n e^{ax} \quad D^n (a^x) = (\log a)^n a^x$$

$$D^n ((ax + b)^m) = \frac{a^n m! (ax + b)^{m-n}}{(m-n)!}, \quad m > n$$

$$D^n \left(\frac{1}{ax + b} \right) = \frac{(-1)^n n! a^n}{(ax + b)^{n+1}} \quad D^n (ax + b)^n = n! a^n,$$

$$D^n (\log(ax + b)) = \frac{(-1)^{n-1} (n-1)! a^n}{(ax + b)^n}$$

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$$D^n(\sin(ax+b)) = a^n \sin(ax+b+n(\pi/2))$$

$$D^n(\cos(ax+b)) = a^n \cos(ax+b+n(\pi/2))$$

$$D^n(e^{ax} \sin(bx+c)) = (a^2+b^2)^{n/2} e^{ax} \sin(bx+c+n \tan^{-1}(b/a))$$

$$D^n(e^{ax} \cos(bx+c)) = (a^2+b^2)^{n/2} e^{ax} \cos(bx+c+n \tan^{-1}(b/a))$$

Worked Examples

I. Find the n^{th} derivatives of the following

$$(1) \log(x^2-9)$$

$$(2) \log \sqrt{\frac{(x+1)(2x+3)}{(3x-4)(x+3)}}$$

$$(3) \log[(3x+5)^2 e^{\cos(3x-2)}]$$

$$(4) \log_{10} \left[\frac{(x+5)(3x+1)^2}{(5x+9)^3 (x-1)^{10}} \right]$$

► (1) Let $y = \log(x^2-9) = \log[(x-3)(x+3)]$

$$y = \log(x-3) + \log(x+3)$$

Differentiating n times, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{(x-3)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+3)^n}$$

$$y_n = (-1)^{n-1}(n-1)! \left\{ \frac{1}{(x-3)^n} + \frac{1}{(x+3)^n} \right\}$$

(2) Let $y = \log \sqrt{\frac{(x+1)(2x+3)}{(3x-4)(x+3)}}$

$$= \log \left(\frac{(x+1)(2x+3)}{(3x-4)(x+3)} \right)^{1/2}$$

$$= \frac{1}{2} \log \left[\frac{(x+1)(2x+3)}{(3x-4)(x+3)} \right]$$

$$y = \frac{1}{2} \{ \log(x+1) + \log(2x+3) - \log(3x-4) - \log(x+3) \}$$

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Differentiating n times, we get

$$y_n = \frac{1}{2} \left\{ \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} + \frac{(-1)^{n-1}(n-1)!2}{(2x+3)^n} - \frac{(-1)^{n-1}(n-1)!3^n}{(3x-4)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+3)^n} \right\}$$

$$= \frac{1}{2}(-1)^{n-1}(n-1)! \left\{ \frac{1}{(x+1)^n} + \frac{2^n}{(2x+3)^n} - \frac{3^n}{(3x-4)^n} - \frac{1}{(x+3)^n} \right\}$$

(3) Let $y = \log[(3x+5)^2 e^{\cos(3x-2)}]$

$$= \log(3x+5)^2 + \log e^{\cos(3x-2)}$$

$$= 2\log(3x+5) + \cos(3x-2)\log e$$

$$y = 2\log(3x+5) + \cos(3x-2) \quad [\because \log_e e = 1]$$

Differentiating n times, we get

$$y_n = 2 \frac{(-1)^{n-1}(n-1)!3^n}{(3x+5)^n} + 3^n \cos\left(3x-2 + \frac{n\pi}{2}\right)$$

(4) Let $y = \log_{10} \left[\frac{(x+5)(3x+1)^2}{(5x+9)^3(x-1)^{10}} \right]$

$$= \frac{\log \left[\frac{(x+5)(3x+1)^2}{(5x+9)^3(x-1)^{10}} \right]}{\log 10} \quad \because \log_a N = \frac{\log N}{\log a}$$

$$= \frac{1}{\log 10} \{ \log(x+5) + \log(3x+1)^2 - \log(5x+9)^3 - \log(x-1)^{10} \}$$

$$y = \frac{1}{\log 10} \{ \log(x+5) + 2.\log(3x+1) - 3.\log(5x+9) - 10.\log(x-1) \}$$

Differentiating n times, we get

$$y_n = \frac{1}{\log 10} \left\{ \frac{(-1)^{n-1}(n-1)!}{(x+5)^n} + 2 \frac{(-1)^{n-1}(n-1)!3^n}{(3x+1)^n} - 3 \frac{(-1)^{n-1}(n-1)!5^n}{(5x+9)^n} - 10 \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} \right\}$$

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$$y_n = \frac{(-1)^{n-1}(n-1)!}{\log 10} \left\{ \frac{1}{(x+5)^n} + \frac{2 \times 3^n}{(3x+1)^n} - \frac{3 \times 5^n}{(5x+9)^n} - \frac{10}{(x-1)^n} \right\} \blacksquare$$

II. Find the n^{th} derivatives of the following

- | | |
|------------------------------|-------------------------|
| (1) $\sin^2 5x + \cos^2 3x$ | (2) $\cos 2x \cos 3x$ |
| (3) $\sin 3x \cos 2x$ | (4) $\sin 4x \sin 3x$ |
| (5) $\cos x \cos 4x \cos 7x$ | (6) $\cos^3 x \sin^3 x$ |
| (7) $e^{2x} \cos^3 x$ | (8) $\sinh x \cos x$ |

► (1) Let $y = \sin^2 5x + \cos^2 3x$

$$\begin{aligned} &= \frac{1}{2}[1 - \cos 10x] + \frac{1}{2}[1 + \cos 6x] \quad \left[\because \sin^2 x = \frac{1}{2}(1 - \cos 2x) \right] \\ &= \frac{1}{2} - \frac{1}{2} \cos 10x + \frac{1}{2} + \frac{1}{2} \cos 6x \quad \left[\because \cos^2 x = \frac{1}{2}(1 + \cos 2x) \right] \\ y &= 1 - \frac{1}{2} \cos 10x + \frac{1}{2} \cos 6x \end{aligned}$$

Differentiating n times, we get

$$y_n = -\frac{10^n}{2} \cos\left(10x + \frac{n\pi}{2}\right) + \frac{6^n}{2} \cos\left(6x + \frac{n\pi}{2}\right)$$

(2) Let $y = \cos 2x \cos 3x = \cos 3x \cos 2x$

$$\begin{aligned} &= \frac{1}{2} \{ \cos(3x+2x) + \cos(3x-2x) \} \\ &\left[\text{By using } \cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \right] \\ &= \frac{1}{2} [\cos 5x + \cos x] \\ y &= \frac{1}{2} \cos 5x + \frac{1}{2} \cos x \end{aligned}$$

Differentiating n times, we get

$$y_n = \frac{5^n}{2} \cos\left(5x + \frac{n\pi}{2}\right) + \frac{1}{2} \cos\left(x + \frac{n\pi}{2}\right)$$

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$$(3) \text{ Let } y = \sin 3x \cos 2x$$

$$= \frac{1}{2} \{ \sin(3x + 2x) + \sin(3x - 2x) \}$$

$$\left[\text{By using } \sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)] \right]$$

$$y = \frac{1}{2} [\sin 5x + \sin x]$$

$$y = \frac{1}{2} \sin 5x + \frac{1}{2} \sin x$$

Differentiating n times, we get

$$y_n = \frac{5^n}{2} \sin \left(5x + \frac{n\pi}{2} \right) + \frac{1}{2} \sin \left(x + \frac{n\pi}{2} \right)$$

$$(4) \text{ Let } y = \sin 4x \sin 3x$$

$$= \frac{1}{2} \{ \cos(4x - 3x) - \cos(4x + 3x) \}$$

$$\left[\text{By using } \cos A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \right]$$

$$y = \frac{1}{2} [\cos x - \cos 5x]$$

$$y = \frac{1}{2} \cos x - \frac{1}{2} \cos 5x$$

Differentiating n times, we get

$$y^n = \frac{1}{2} \cos \left(x + \frac{n\pi}{2} \right) - \frac{5^n}{2} \cos \left(5x + \frac{n\pi}{2} \right)$$

$$(5) \text{ Let } y = \cos x \cos 4x \cos 7x$$

$$= \cos x (\cos 7x \cos 4x)$$

$$= \cos x \left[\frac{1}{2} \{ \cos(7x + 4x) + \cos(7x - 4x) \} \right]$$

$$\left[\text{By using } \cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)] \right]$$

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$$\begin{aligned}
 y &= \frac{1}{2} \cos x [\cos 11x + \cos 3x] \\
 &= \frac{1}{2} \cos 11x \cos x + \frac{1}{2} \cos 3x \cos x \\
 &= \frac{1}{4} [\cos 12x + \cos 10x] + \frac{1}{4} [\cos 4x + \cos 2x] \\
 y &= \frac{1}{4} \cos 12x + \frac{1}{4} \cos 10x + \frac{1}{4} \cos 4x + \frac{1}{4} \cos 2x
 \end{aligned}$$

Differentiating n times, we get

$$\begin{aligned}
 y_n &= \frac{12^n}{4} \cos \left(12x + \frac{n\pi}{2} \right) + \frac{10^n}{4} \cos \left(10x + \frac{n\pi}{2} \right) \\
 &\quad + \frac{4^n}{4} \cos \left(4x + \frac{n\pi}{2} \right) + \frac{2^n}{4} \cos \left(2x + \frac{n\pi}{2} \right)
 \end{aligned}$$

(6) Let $y = \cos^3 x \sin^3 x$

$$y = (\cos x \sin x)^3 \quad [\because \sin 2x = 2 \sin x \cos x]$$

$$= \left[\frac{\sin 2x}{2} \right]^3 \quad [\because \sin^3 A = \frac{1}{4} \{3 \sin A - \sin 3A\}]$$

$$= \frac{\sin^3 2x}{8} = \frac{1}{8} \left[\frac{1}{4} \{3 \sin 2x - \sin 6x\} \right]$$

$$= \frac{1}{32} [3 \sin 2x - \sin 6x]$$

$$y = \frac{3}{32} \sin 2x - \frac{1}{32} \sin 6x$$

Differentiating n times, we get

$$y_n = \frac{3 \times 2^n}{32} \sin \left(2x + \frac{n\pi}{2} \right) - \frac{6^n}{32} \sin \left(6x + \frac{n\pi}{2} \right)$$

(7) Let $y = e^{2x} \cos^3 x$

$$y = e^{2x} \left\{ \left(\frac{1}{4} \right) [3 \cos x + \cos 3x] \right\}$$

$$[\because \cos^3 A = \left(\frac{1}{4} \right) [3 \cos A + \cos 3A]]$$

$$y = \frac{3}{4} e^{2x} \cos x + \frac{1}{4} e^{2x} \cos 3x$$

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Differentiating n times, we get

$$y_n = \frac{3}{4} \left\{ (2^2 + 1^2)^n e^{2x} \cos \left(x + n \tan^{-1} \left(\frac{1}{2} \right) \right) \right\}$$

$$+ \frac{1}{4} \left\{ (2^2 + 3^2)^n e^{2x} \cos \left(3x + n \tan^{-1} \left(\frac{3}{2} \right) \right) \right\}$$

$$y_n = \frac{3 \times 5^2}{4} e^{2x} \cos \left(x + n \tan^{-1} \left(\frac{1}{2} \right) \right) + \frac{13^2}{4} e^{2x} \cos \left(3x + n \tan^{-1} \left(\frac{3}{2} \right) \right)$$

(8) Let $y = \sinh x \cos x = \left(\frac{e^x - e^{-x}}{2} \right) \cos x = \frac{1}{2} e^x \cos x - \frac{1}{2} e^{-x} \cos x$

Differentiating n times, we get

$$y_n = \frac{1}{2} \left[(1^2 + 1^2)^n e^x \cos \left(x + \frac{n\pi}{4} \right) \right]$$

$$- \frac{1}{2} \left[((-1)^2 + 1^2)^n e^{-x} \cos \left(x + \frac{3n\pi}{4} \right) \right]$$

$$y_n = \frac{2^2}{2} e^x \cos \left(x + \frac{n\pi}{4} \right) - \frac{2^2}{2} e^{-x} \cos \left(x + \frac{3n\pi}{4} \right) \quad \blacksquare$$

III. Find the n^{th} derivatives of the following

(1) $\frac{6x}{(x^2-4)(x-1)}$ (2) $\frac{x}{(x-1)(2x+3)}$

(3) $\frac{x}{x^2-5x+6}$ (4) $\frac{x^2}{(2x+1)(2x+3)}$

► (1) Let $y = \frac{6x}{(x^2-4)(x-1)}$

$$\text{Let } \frac{6x}{(x^2-4)(x-1)} = \frac{6x}{(x-2)(x+2)(x-1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$\Rightarrow 6x = A(x+2)(x-1) + B(x-2)(x-1) + C(x-2)(x+2)$$

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$$\text{Put } x = 1 \Rightarrow 6 = -3C \Rightarrow C = -2,$$

$$\text{Put } x = 2 \Rightarrow 12 = 4A \Rightarrow A = 3,$$

$$\text{Put } x = -2 \Rightarrow -12 = 12B \Rightarrow B = -1$$

$$\therefore \frac{6x}{(x^2-4)(x-1)} = \frac{3}{x-2} + \frac{-1}{x+2} + \frac{-2}{x-1}$$

$$\therefore y = \frac{3}{x-2} - \frac{1}{x+2} - \frac{2}{x-1}$$

Differentiating n times, we get

$$y_n = \frac{3(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}} - \frac{2(-1)^n n!}{(x-1)^{n+1}}$$

$$(2) \text{ Let } y = \frac{x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$$

$$\Rightarrow x = A(2x+3) + B(x-1)$$

$$\text{Put } x = 1 \Rightarrow 1 = 5A \Rightarrow A = \frac{1}{5}$$

$$\text{Put } x = -\frac{3}{2} \Rightarrow -\frac{3}{2} = -\frac{5}{2}B \Rightarrow B = \frac{3}{5}$$

$$\therefore \frac{x}{(x-1)(2x+3)} = \frac{1}{5(x-1)} + \frac{3}{5(2x+3)}$$

$$\therefore y = \frac{1}{5(x-1)} + \frac{3}{5(2x+3)}$$

Differentiating n times, we get

$$y_n = \frac{1(-1)^n n!}{5(x-1)^{n+1}} + \frac{3(-1)^n n 2^n}{5(2x+3)^{n+1}}$$

$$(3) \text{ Let } y = \frac{x}{x^2-5x+6}$$

$$\text{Let } \frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\Rightarrow x = A(x-3) + B(x-2)$$

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$$\text{Put } x = 3 \Rightarrow B = 3,$$

$$\text{Put } x = 2 \Rightarrow -A = 2 \Rightarrow A = -2$$

$$\therefore y = \frac{-2}{x-2} + \frac{3}{x-3} = \frac{3}{x-3} - \frac{2}{x-2}$$

Differentiating n times, we get

$$y_n = \frac{3(-1)^n n!}{(x-3)^{n+1}} - \frac{2(-1)^n n!}{(x-2)^{n+1}}$$

$$(4) \text{ Let } y = \frac{x^2}{(2x+1)(2x+3)}$$

This is an improper fraction, converting this into proper fractions, we get

$$y = \frac{x^2}{(2x+1)(2x+3)} = \frac{x^2}{4x^2 + 8x + 3}$$

$$\frac{4x^2 + 8x + 3}{4x^2 + 8x + 3} \times \frac{x^2}{4x^2 + 8x + 3} = \frac{(1/4)x^2}{4x^2 + 8x + 3}$$

$$\frac{x^2 + 2x + 3/4}{-2x - 3/4}$$

$$\frac{x^2}{4x^2 + 8x + 3} = \frac{1}{4} + \frac{-2x - 3/4}{4x^2 + 8x + 3}$$

$$\therefore y = \frac{1}{4} - \frac{2x + 3/4}{(2x+1)(2x+3)} \quad \text{---(1)}$$

$$\text{Let } \frac{2x + 3/4}{(2x+1)(2x+3)} = \frac{A}{2x+1} + \frac{B}{2x+3}$$

$$\Rightarrow 2x + (3/4) = A(2x+3) + B(2x+1)$$

$$\text{Put } x = -\frac{3}{2} \Rightarrow 2\left(-\frac{3}{2}\right) + \frac{3}{4} = B\left[2\left(-\frac{3}{2}\right) + 1\right]$$

$$\Rightarrow -\frac{9}{4} = -2B \Rightarrow B = \frac{9}{8}$$

$$\text{Put } x = -\frac{1}{2} \Rightarrow 2\left(-\frac{1}{2}\right) + \frac{3}{4} = A\left[2\left(-\frac{1}{2}\right) + 3\right]$$

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$$\Rightarrow -\frac{1}{4} = 2A \Rightarrow A = -\frac{1}{8}$$

$$\frac{2x+3/4}{(2x+1)(2x+3)} = \frac{-1/8}{2x+1} + \frac{9/8}{2x+3}$$

$$\frac{2x+3/4}{(2x+1)(2x+3)} = \frac{9}{8} \frac{1}{(2x+3)} - \frac{1}{8} \frac{1}{(2x+1)} \quad \text{---(2)}$$

Substituting (2) in (1), we get

$$y = \frac{1}{4} - \left\{ \frac{9}{8} \frac{1}{(2x+3)} - \frac{1}{8} \frac{1}{(2x+1)} \right\}$$

$$y = \frac{1}{4} + \frac{1}{8} \frac{1}{(2x+1)} - \frac{9}{8} \frac{1}{(2x+3)}$$

Differentiating n times, we get

$$y_n = \frac{1}{8} \frac{(-1)^n n! 2^n}{(2x+1)^{n+1}} - \frac{9}{8} \frac{(-1)^n n! 2^n}{(2x+3)^{n+1}} \quad \blacksquare$$

IV. Prove the following

(1) $D^n \left(\frac{1}{x^2 + a^2} \right) = \frac{(-1)^n n!}{a^{n+1}} \sin(n+1)\theta$,
 where $r = \sqrt{x^2 + a^2}$, $\theta = \tan^{-1}(x/a)$

(2) $D^n \left(\tan^{-1} \frac{x}{a} \right) = \frac{(-1)^{n+1} (n-1)!}{r^n} \sin n\theta$,
 where $r = \sqrt{x^2 + a^2}$, $\theta = \tan^{-1}(x/a)$

(3) $D^n \left(\tanh^{-1} \frac{x}{a} \right) = \frac{(n-1)!}{2} \left[\frac{1}{(a-x)^n} + \frac{(-1)^{n-1}}{(a+x)^n} \right]$

➔ (1) Let $y = \frac{1}{x^2 + a^2} = \frac{1}{x^2 - i^2 a^2} = \frac{1}{x^2 - (ia)^2}$

$$y = \frac{1}{(x+ai)(x-ai)} \quad \text{---(1)}$$

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$$\text{Let } \frac{1}{(x+ai)(x-ai)} = \frac{A}{x+ai} + \frac{B}{x-ai}$$

$$\Rightarrow 1 = A(x-ai) + B(x+ai)$$

$$\therefore \text{ Put } x = ai \Rightarrow 1 = 2aiB \Rightarrow B = \frac{1}{2ai}$$

$$x = -ai \Rightarrow 1 = -2aiA \Rightarrow A = -\frac{1}{2ai}$$

$$\therefore \frac{1}{(x+ai)(x-ai)} = \frac{-1/2ai}{x+ai} + \frac{1/2ai}{x-ai} \quad \text{---(2)}$$

Substituting (2) in (1), we get

$$y = \frac{1}{2ai} \left\{ \frac{1}{x-ai} - \frac{1}{x+ai} \right\}$$

Differentiating n times, we get

$$y_n = \frac{1}{2ai} \left\{ \frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}} \right\}$$

$$y_n = \frac{(-1)^n n!}{2ai} \left\{ \frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}} \right\} \quad \text{---(3)}$$

Put $x = r \cos \theta$ $a = r \sin \theta$

$$x - ai = r \cos \theta - ir \sin \theta = r(\cos \theta - i \sin \theta)$$

$$(x - ai) = r e^{-i\theta}$$

$$\therefore e^{i\theta} = \cos \theta + i \sin \theta$$

$$(x - ai)^{n+1} = r^{n+1} e^{-i(n+1)\theta}$$

$$\therefore e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\frac{1}{(x - ai)^{n+1}} = \frac{1}{r^{n+1} e^{-i(n+1)\theta}}$$

$$\frac{1}{(x - ai)^{n+1}} = \frac{e^{i(n+1)\theta}}{r^{n+1}} \quad \text{---(4)}$$

$$\text{Similarly, } \frac{1}{(x + ai)^{n+1}} = \frac{e^{-i(n+1)\theta}}{r^{n+1}} \quad \text{---(5)}$$

Substitute (4) and (5) in (3), we get

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$$y_n = \frac{(-1)^n n!}{2ai} \left\{ \frac{e^{i(n+1)\theta}}{r^{n+1}} - \frac{e^{-i(n+1)\theta}}{r^{n+1}} \right\}$$

$$y_n = \frac{(-1)^n n!}{ar^{n+1}} \left\{ \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{2i} \right\}$$

$$D^n \left(\frac{1}{x^2 + a^2} \right) = \frac{(-1)^n n!}{ar^{n+1}} \sin(n+1)\theta \quad \left[\because \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta \right]$$

(2) Let $y = \tan^{-1} \left(\frac{x}{a} \right)$

Differentiating with respect to x , we get

$$y_1 = \frac{1}{1^2 + (x/a)^2} \cdot \frac{1}{a} = \frac{1}{a} \cdot \frac{1}{(a^2 + x^2)/a^2} = \frac{a}{a^2 + x^2}$$

$$y_1 = \frac{1}{2i} \left\{ \frac{1}{x - ai} - \frac{1}{x + ai} \right\} \text{ [by partial fractions] (Refer example 1)}$$

Differentiating $(n-1)$ times, we get

$$D^{n-1}(y_1) = \frac{1}{2i} \left[D^{n-1} \left\{ \frac{1}{x - ai} - \frac{1}{x + ai} \right\} \right]$$

$$y_n = \frac{1}{2i} \left[\frac{(-1)^{n-1} (n-1)!}{(x - ai)^n} - \frac{(-1)^{n-1} (n-1)!}{(x + ai)^n} \right]$$

$$y_n = \frac{(-1)^{n-1} (n-1)!}{2i} \left[\frac{1}{(x - ai)^n} - \frac{1}{(x + ai)^n} \right] \quad \text{---(1)}$$

Put $x = r \cos \theta$ $a = r \sin \theta$

$$x - ai = r \cos \theta - i r \sin \theta = r (\cos \theta - i \sin \theta)$$

$$x - ai = r e^{-i\theta}$$

$$(x - ai)^n = r^n e^{-in\theta}$$

$$\frac{1}{(x - ai)^n} = \frac{1}{r^n e^{-in\theta}} = \frac{e^{in\theta}}{r^n} \quad \text{---(2)}$$

Similarly, $\frac{1}{(x + ai)^n} = \frac{e^{-in\theta}}{r^n} \quad \text{---(3)}$

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