CHAPTER – 1

COMPLEX NUMBERS

In solving algebraic equations, the real number system was known to mathematicians in the past. It therefore became necessary to extend the real number system to obtain necessary solutions to simple equations like $x^2 + 1 = 0$, which could not be solved in the domain of real number. The equation becomes solvable if we introduce new quantities, square roots of negative number. An Italian mathematician Cardano first introduced square roots of negative numbers in the 16th century. Two hundred years later Euler and John Bernoulli recognized the complex numbers and introduced the symbol i for $\sqrt{-1}$. Gauss and Hamilton gave a solid foundation for the theory of complex numbers in 19th century. The concept of complex numbers is a powerful and a widely used tool in mathematics. It is necessary for engineering students to have some knowledge of complex numbers. Here in this chapter we discuss the basics of complex numbers.

Definition

Any number of the form $z = x + iy$, where and x and y are real numbers and $i = \sqrt{-1}$ is called complex number. The number x is called the real part denoted by $R(z)$ or $Re(z)$ and the number y is called imaginary part denoted by $I(z)$ or $Im(z)$ of complex number.

The system of complex numbers is the set of ordered pairs (x, y) of real numbers let C denote the set of all ordered pairs of real numbers.

Thus $C = \{(x, y) : x, y \in R\}$

Note that $i^2 = -1; i^3 = -i, i^4 = 1$ and $i^{4n} = 1$, where n is an integer.

Algebra of Complex Numbers

Addition, Subtraction, Multiplication and Division of Complex Numbers

1. ***Addition***

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

2. *Subtraction*

If $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$, then
$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

3. *Multiplication*

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 \times z_2 = (x_1 + iy_1) \times (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

If k is a real number, then
$kz_1 = k(x_1 + iy_1) = kx_1 + iky_1$

4. *Division*

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$ then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{(x_1 x_2 + y_1 y_2)}{(x_2^2 + y_2^2)} + i\frac{(y_1 x_2 - x_1 y_2)}{(x_2^2 + y_2^2)}$$

Equality of Complex Numbers

Two complex numbers are said to be equal if and only if they are identical i.e., if their respective real and imaginary parts are equal.

Let $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$

If $z_1 = z_2$, then we have

$$x_1 + iy_1 = x_2 + iy_2$$
$$\Rightarrow x_1 - x_2 = iy_2 - iy_1$$
$$\Rightarrow x_1 - x_2 = -i(y_1 - y_2)$$
$$\Rightarrow (x_1 - x_2)^2 = (-i)^2 (y_1 - y_2)^2$$
$$\Rightarrow (x_1 - x_2)^2 = -(y_1 - y_2)^2$$
$$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$$

$$\Rightarrow (x_1 - x_2)^2 = 0 \text{ and } (y_1 - y_2)^2 = 0$$

$$\Rightarrow (x_1 - x_2) = 0 \text{ and } y_1 - y_2 = 0$$

$$\Rightarrow x_1 = x_2 \text{ and } y_1 = y_2$$

This shows that two complex numbers will be equal if and only if their real and imaginary parts are equal.

Conjugate of a Complex Number

If $z = x + iy$ is any complex number, then its conjugate or complex conjugate is denoted and is defined as

$$\bar{z} = x - iy$$

Properties

(i) If $z = x + iy$, then $\bar{z} = x - iy$

(ii) If $z = x - iy$ then $\bar{z} = x + iy$

(iii) $\overline{(\bar{z})} = z$

(iv) $z + \bar{z} = 2x = 2\,\mathrm{Re}(z)$

(v) $z - \bar{z} = 2iy = i2\,\mathrm{Im}(z)$

(vi) $z\bar{z} = x^2 + y^2$, real number

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Geometrical Representation and Polar Form of a Complex Number

Let $P(x, y)$ be a point that represents the complex number $z = x + iy$ referred to rectangular axes OX and OY, usually OX is called the real axis and OY is called the imaginary axis. The plane is called Argand plane or complex plane or Gaussian plane. The point is called the image of the complex number z and z is said to be affix or complex co-ordinate of point P.

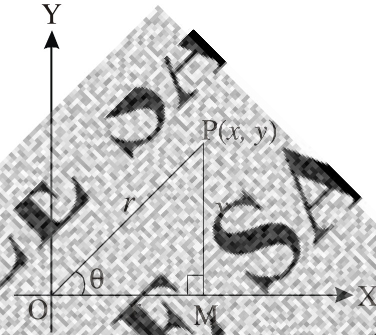

Figure 1.1

Let $OP = r$, $X\hat{O}P = \hat{\theta}$, then $OP^2 = OM^2 + PM^2$

$$\Rightarrow r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

Also $\cos\theta = \frac{OM}{OP} = \frac{x}{r} \Rightarrow x = r\cos\theta$

and $\sin\theta = \frac{PM}{OP} = \frac{y}{r} \Rightarrow y = r\sin\theta$

Therefore, $z = x + iy = r(\cos\theta + i\sin\theta)$

The representation of complex number in the form $r(\cos\theta + i\sin\theta)$ is called the polar form and $z = x + iy$ is called the rectangular form or Cartesian form.

Geometrical Representation of Algebraic Operation on Complex Number

Sum (Addition)

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively. By the definition $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ should be represented by

$$P(x_1 + x_2, y_1 + y_2)$$

Let P_1L, P_2N and PK be parallel to the y – axis and P_1M is parallel to x-axis. Note that triangles OP_2N and PP_1M are congruent.

We have $ON = P_1M = x_2$, $P_2N = PM = y_2$

Therefore, $OK = OL + LK = OL + P_1M = x_1 + x_2$

$$PK = PM + MK = P_2N + P_1L = y_2 + y_1$$

Hence, the co-ordinates of the point P

$$\left(x_1 + x_2,\ y_1 + y_2\right)$$

Thus this shows that the point P is nothing but the rate x which complete the parallelogram with the line segments joining the origin with the z_1 and z_2 as the adjacent sides.

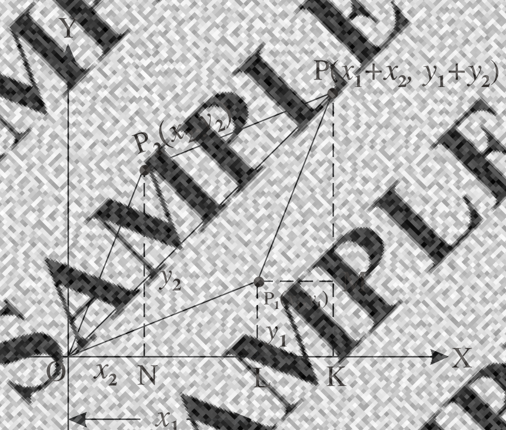

Figure 1.2

Difference (Subtraction)

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ to be two complex numbers represented by the points $P(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively. By the definition $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)$ be represented by $(x_1 - x_2,\ y_1 - y_2)$

Let us denote $-z_2$ by the point P_2' so that P_2P_2' is bisected at O complete the parallelogram OP_1PP_2'. Then it can be easily seen that P represents the difference $z_1 - z_2$.

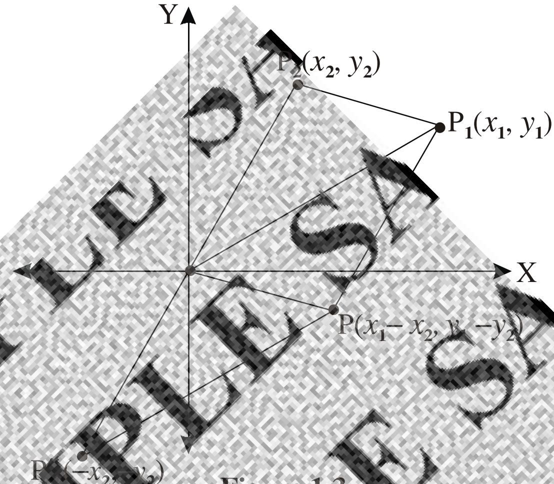

Figure 1.3

3. Product

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 + i\sin\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i^2 \sin\theta_1 \sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 + i\sin\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 - \sin\theta_1 \sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2))$$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

thus $|z_1 z_2| = r_1 r_2$ and $\text{amp}(z_1 z_2) = \theta_1 + \theta_2$

This shows that a module of the product of two complex numbers is product of their moduli and the argument (amplitude) is the sum of their arguments.

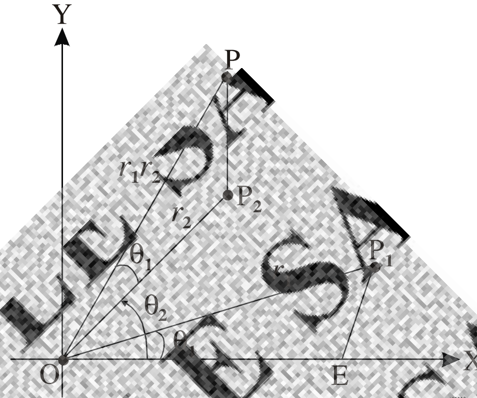

Figure 1.4

Let P_1 and P_2 be two points represented by the complex numbers z_1 and z_2 respectively. Let E be the point on the x–axis such that $OE = 1$ unit. Complete the triangle OP_1E. Now taking OP_2 as the base, construct a triangle OPP_2 Similar to OP_1E so that

$$OP : OP_1 = OP_2 : OE$$

$$OP.OE = OP_1.OP_2$$

$$\therefore OP = OP_1.OP_2 \quad (\because OE = 1 \text{ Unit})$$

also $P_2\hat{O}P = X\hat{O}P_1 = \theta_1$

thus $X\hat{O}P = \theta_1 + \theta_2$

Hence, P represents the complex number for which modulus is $r_1 r_2$ and amplitude is $\theta_1 + \theta_2$, That is, P represents the complex number $z_1 z_2$

Division

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

We take $z_2 \neq 0$ so that $r_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

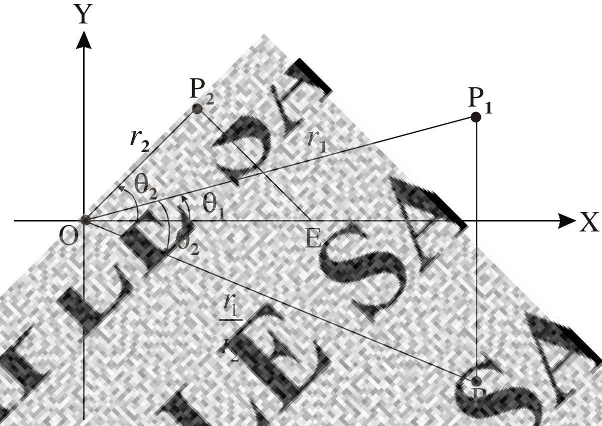

Figure 1.5

Let P_1 and P_2 be two points represent z_1 and z_2 respectively. On OP_1 construct the triangle OPP_1 similar to OEP_2. Where E it is the point an the x - axis and $OE = 1$ unit

Since OEP_2 and OPP_1 are similar triangles,

$$\therefore OP : OE = r_1 : r_2$$

$$r_2 . OP = r_1 . OE$$

$$r_2 . OP = r_1 \qquad (\because OE = 1 \text{Unit})$$

$$OP = \frac{r_1}{r_2}$$

also $X\hat{O}P = \theta_1 - \theta_2$

Thus the point P represents the quotient $\frac{z_1}{z_2}$ since its modulus $\frac{r_1}{r_2}$ and amplitude $\theta_1 - \theta_2$

Modulus and amplitude of a complex number

If $z = x + iy$ is a complex number, then its modulus or absolute value is denoted by $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2}$$

where $|z|$ is a non negative real number i.e., $|z| \geq 0$

It can be proved that (i) $|\bar{z}| = |z|$ (ii) $z\bar{z} = |z|^2$

(iii) $|z_1 \cdot z_2| = |z_1||z_2|$ (iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

The amplitude of the complex number is denoted by $\text{amp}\, z$ or $\arg(z)$ and is defined as

$$\text{amp}\, z = \tan^{-1}\left(\frac{y}{x}\right)$$

We know that $\tan^{-1}\left(\frac{y}{x}\right)$ has many values. The smallest numerical value falling in the quadrant of the complex number is called the fundamental amplitude. The general value $2n\pi + \tan^{-1}\left(\frac{y}{x}\right)$ of the fundamental amplitude is called the general amplitude.

Method of calculating fundamental amplitude

Let $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

If z is in the first quadrant then amp $z = \alpha$
If z is in the second quadrant then amp $z = \pi - \alpha$
If z is in the third quadrant then amp $z = \pi + \alpha$
If z is in the fourth quadrant then amp $z = 2\pi - \alpha$

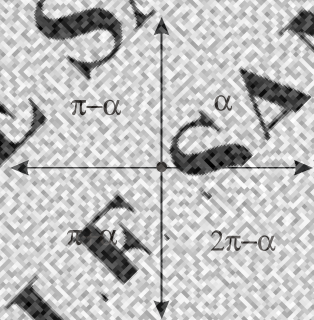

Figure 1.6

Illustrations

1. $z = \sqrt{3} + i$

$$\alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$$

Since $z = \sqrt{3} + i = (\sqrt{3}, 1)$ is in first quadrant. Therefore, $\text{amp}\, z = \alpha = \frac{\pi}{6}$

2. $z = -\sqrt{3} + i$

$$\alpha = \tan^{-1}\left|\frac{1}{-\sqrt{3}}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Since $z = -\sqrt{3} + i = (-\sqrt{3}, 1)$ is in second quadrant

Therefore,

$$\text{amp}\, z = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

3. $z = -\sqrt{3} - i$

$$\alpha = \tan^{-1}\left|\frac{-1}{-\sqrt{3}}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Since $z = -\sqrt{3} - i = (-\sqrt{3}, -1)$ is in the third quadrant

Therefore,

$$\text{amp}\, z = \pi + \alpha = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

4. $z = \sqrt{3} - i$

$$\alpha = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Since $z = \sqrt{3} - i = (\sqrt{3}, -1)$ is in fourth quadrant

Therefore,

$$\text{amp}\, z = 2\pi - \alpha = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Properties

(i) $\text{amp}(z_1 \cdot z_2) = \text{amp}z_1 + \text{amp}z_2$

(ii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}z_1 - \text{amp}z_2$

(iii) $\text{amp}\ z^2 = 2\text{amp}z$

(iv) $\text{amp}\ z^n = n\text{amp}z$

Exponential Form

We have

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - + \ldots\ldots\ldots\ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \ldots\ldots\ldots\ldots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$$

Therefore, $e^{ix} = 1 + \dfrac{ix}{1!} + \dfrac{i^2x^2}{2!} + \dfrac{i^3x^3}{3!} + \dfrac{i^4x^4}{4!} + \ldots\ldots\ldots$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \ldots\ldots\ldots\right) + i\left(\frac{x}{1!} - \frac{x^3}{3!} + - \ldots\ldots\ldots\right)$$

$= \cos x + i\sin x$, which is known as Euler's formula.

Now, $x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$, which is known as exponential form of the complex number $x + iy$.

Worked Examples

1. *Express the following in the form of* $x + iy$

(i) $\dfrac{(1+i)}{(1-i)}$ (ii) $\dfrac{(3+i)(1-3i)}{2+i}$ (iii) $\dfrac{1}{3+2i}$

(iv) $\dfrac{3+4i}{3-4i}$ (v) $\dfrac{5+5i}{3-4i} + \dfrac{20}{4+3i}$ (vi) $\dfrac{3i^{30} - i^{19}}{2i-1}$

➡ (i) Let $z = \dfrac{1+i}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i}$

$$=\frac{(1+i)^2}{1^2-i^2}=\frac{1^2+i^2+2.1.i}{1-(-1)}=\frac{1-1+2i}{2}=i$$

(ii) Let $z=\dfrac{(3+i)(1-3i)}{2+i}$

$$=\frac{3-9i+i-3i^2}{2+i}=\frac{3-8i+3}{2+i}$$

$$=\frac{6-8i}{2+i}\times\frac{2-i}{2-i}=\frac{12-6i-16i+8i^2}{2^2-i^2}$$

$$=\frac{12-22i-8}{4+1}=\frac{4}{5}-\frac{22}{5}i$$

(iii) Let $z=\dfrac{1}{3+2i}=\dfrac{1}{3+2i}\times\dfrac{3-2i}{3-2i}$

$$=\frac{3-2i}{9-4i^2}=\frac{3-2i}{13}=\frac{3}{13}-\frac{2}{13}i$$

(iv) Let $z=\dfrac{3+4i}{3-4i}=\dfrac{3+4i}{3-4i}\times\dfrac{3+4i}{3+4i}$

$$=\frac{(3+4i)^2}{9-16i^2}=\frac{9+16i^2+24i}{25}=\frac{-7+24i}{25}$$

$$z=\frac{-7}{25}+\frac{24}{25}i$$

(v) Let $z=\dfrac{5+5i}{3-4i}+\dfrac{20}{4+3i}$

$$=\frac{5+5i}{3-4i}\times\frac{3+4i}{3+4i}+\frac{20}{4+3i}\times\frac{4-3i}{4-3i}$$

$$=\frac{15+15i+20i+20i^2}{9-16i^2}+\frac{80-60i}{16-9i^2}$$

$$=\frac{-5+35i}{25}+\frac{80-60i}{25}$$

$$=\frac{-5+35i+80-60i}{25}=\frac{75-25i}{25}=3-i$$

(vi) $\dfrac{3i^{30}-i^{19}}{2i-1}=\dfrac{3\left(i^2\right)^{15}-\left(i^2\right)^9 i}{2i-1}$

$$=\frac{3(-1)^{15}-(-1)^9 i}{2i-1}=\frac{-3+i}{2i-1}\times\frac{2i+1}{2i+1}$$

$$=\frac{-6i+2i^2-3+i}{4i^2-1}=\frac{5i-5}{-5}$$

$$=-i+1$$

■

2. *If* $z_1=2+i,\quad z_2=3-2i$ *and* $z_3=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, *evaluate the following*

(i) $|3z_1-4z_2|$, (ii) z_3^4 and (iii) $\left|\dfrac{2z_2+z_1-5-i}{2z_1-z_2+3-i}\right|$

➡ (i) $|3z_1-4z_2|=|3(2+i)-4(3-2i)|$

$$=|6+3i-12+8i|=|-6+11i|$$

$$=\sqrt{(-6)^2+11^2}=\sqrt{157}$$

(ii) $z_3^4=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^4$

$$=\left[\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2\right]^2=\left[\frac{1}{4}+\frac{3}{4}i^2-2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}i\right]^2$$

$$=\left[\frac{1}{4}-\frac{3}{4}-\frac{\sqrt{3}}{2}i\right]^2=\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right]^2=\left[-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]^2$$

$$=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{1}{4}+\frac{3}{4}i^2+2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}i$$

$$=\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}}{2}i$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

(iii) $\left|\dfrac{2z_2+z_1-5-i}{2z_1-z_2+3-i}\right| = \left|\dfrac{2(3-2i)+(2+i)-5-i}{2(2+i)-(3-2i)+3-i}\right|$

$$= \left|\frac{6-4i+2+i-5-i}{4+2i-3+2i+3-i}\right|$$

$$= \left|\frac{3-4i}{4+3i}\right| = \frac{\sqrt{3^2+(-4)^2}}{\sqrt{4^2+3^2}}$$

$$= \frac{\sqrt{25}}{\sqrt{25}} = 1$$ ■

3. *Express the following in the form $a+ib$ and plot them on a complex plane (or organd plane)*

(i) $\dfrac{(2+i)^2}{1+i}$ (ii) $\dfrac{3+i}{(4-2i)(1+i)}$ (iii) $\dfrac{1}{(1-i)^2}-\dfrac{1}{(1+i)^2}$

➡ (i) Let $z=\dfrac{(2+i)^2}{1+i}=\dfrac{4+i^2+4i}{1+i}=\dfrac{4-1+4i}{1+i}=\dfrac{3+4i}{1+i}\times\dfrac{1-i}{1-i}$

$$=\frac{3+4i-3i-4i^2}{1^2-i^2}$$

$$z=\frac{3+i+4}{1+1}=\frac{7}{2}+\frac{1}{2}i=\left(\frac{7}{2},\frac{1}{2}\right)\text{ say}$$

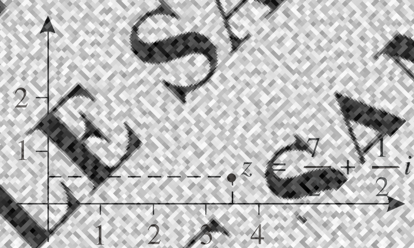

Figure [illegible]

(ii) Let $z=\dfrac{3+i}{(4-2i)(1+i)}=\dfrac{3+i}{4-2i+4i-2i^2}$