15. *Solve* $(D^2+9)(D^2+16)y=\sin 3x$

➡ The AE is $(m^2+9)(m^2+16)=0$

$$m=\pm 3i,\ \pm 4i$$

Therefore, C.F. $=c_1\cos 3x+c_2\sin 3x+c_3\cos 4x+c_4\sin 4x$

$$\text{P.I.}=\frac{\sin 3x}{(D^2+9)(D^2+16)}=\frac{\sin 3x}{(D^2+9)(-3^2+16)}$$

$$=\frac{1}{7}\frac{\sin 3x}{(D^2+9)}=\frac{1}{7}\left\{x\cdot\frac{\sin 3x}{2D}\right\}$$

$$=\frac{x}{14}\int \sin 3x\,dx=\frac{x}{14}\left(-\frac{\cos 3x}{3}\right)=\frac{-x}{42}\cos 3x$$

Hence, the general solution is

$$y=c_1\cos 3x+c_2\sin 3x+c_3\cos 4x+c_4\sin 4x-\frac{x}{42}\cos 3x \quad ■$$

Exercises

1) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=2\sin 3x$

2) $\frac{d^2y}{dx^2}+4y=e^x+\sin 2x$

3) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x$

4) $y''-4y'+4y=\sin 2x+\cos 2x$

5) $(D^2-4)y=8\cos^2 x$

6) $(D^2+4)y=\sin^2 x$

7) $(D^2-4)y=\cos x\cos 2x$

8) $(D^2-4D+3)y=\sin 3x\cos 2x$

9) $(D^2+4)y=\sin^3 x-\cos^2 x$

10) $(D^2+16)y=e^{-3x}+\cos 4x$

11) $(D^3+D^2+D+1)y=\sin 3x$

12) $(D^2+1)(D^2+4)y=\cos 2x+\sin x$

13) $(D^2-D-2)y=2^{x+3}+\cos x$

14) $(D^2+4)(D^2-1)y=\sin 3x+e^{x}$

15) $D^2(D^2+1)(D^2+2)y=\sin 2x$

16) $(D^3+1)y=\cos(2x-1)$

17) $(D^2+16)y=\sin 5x\cos 3x$

18) $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$

19) $(D^4 - 2D^2 + 1)y = \cos x$

20) $(D^4 - 1)y = \cos x$

21) $(D^2 + 4D + 5)y = \sin x$

22) $(D^2 - 3D - 4)y = 10\cos 2x$

Answers

1) $y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{15}(2\cos 3x + \sin 3x)$

2) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{x}{4}\cos 2x$

3) $y = e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] - \frac{1}{13}(2\cos 2x + 3\sin 2x)$

4) $y = (c_1 + c_2 x)e^{2x} + \frac{1}{8}\cos 2x - \frac{1}{8}\sin 2x$

5) $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{2}(2 + \cos 2x)$

6) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(1 - x\sin 2x)$

7) $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{2}\left(\frac{1}{13}\cos 3x + \frac{1}{5}\cos x\right)$

8) $y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$

9) $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{3}{20}\sin x + \frac{1}{52}\sin 3x + \frac{1}{8} + \frac{1}{16}\cos 2x$

10) $y = c_1 \cos 4x + c_2 \sin 4x + \frac{e^{-3x}}{25} + \frac{x}{8}\sin 4x$

11) $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{80}(3\cos 3x - \sin 3x)$

12) $y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x - \frac{x}{12}\sin 2x - \frac{x}{6}\cos x$

13) $y = c_1 e^{-x} + c_2 e^{2x} + \frac{2^{x+3}}{(\log 2)^2 - \log 2 - 2}$

14) $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x + \dfrac{\sin 3x}{50} + \dfrac{xe^{x}}{10}$

15) $y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x + c_5 \cos\left(\sqrt{2}x\right)$
$$+ c_6 \sin\left(\sqrt{2}x\right) - \frac{1}{24}\sin 2x$$

16) $y = c_1 e^{-x} + e^{-\frac{x}{2}}\left[c_2 \cos\left(\dfrac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\dfrac{\sqrt{3}}{2}x\right)\right]$
$$+ \frac{1}{65}\cos(2x-1) - \frac{8}{65}\sin(2x-1)$$

17) $y = c_1 \cos 4x + c_2 \sin 4x + \dfrac{1}{24}\sin 2x - \dfrac{1}{96}\sin 8x$

18) $y = c_1 e^{x} + e^{x}(c_2 \cos x + c_3 \sin x) + xe^{x} + \dfrac{1}{10}(3\sin x + \cos x)$

19) $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)e^{-x} + \dfrac{1}{4}\cos x$

20) $y = c_1 \cos x + c_2 \sin x + c_3 e^{x} + c_4 e^{-x} - \dfrac{x}{4}\sin x$

21) $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \dfrac{1}{8}(\sin x - \cos x)$

22) $y = c_1 e^{-x} + c_2 e^{4x} - \dfrac{4}{5}\cos 2x - \dfrac{3}{5}\sin 2x$

Particular Integral when $f(x)$ is a polynomial

Let $f(x) = g_m(x)$ a polynomial of degree *m*.

In this case P.I. is given by

$$\text{P.I.} = \frac{1}{F(D)}\{g_m(x)\}. \qquad \text{---(1)}$$

To evaluate RHS of (1), we write $\dfrac{1}{F(D)}$ as $\dfrac{1}{1 \pm \phi(D)} = [1 \pm \phi(D)]^{-1}$ by taking least degree term common factor in $F(D)$ and expand it in ascending powers of D by Binomial series expansion i.e., $[1 \pm \phi(D)]^{-1} = \{a_0 + a_1 D + a_2 D^2 + ---\}$ ---(2)

Then (1) becomes, P.I. $= \{a_0 + a_1 D + a_2 D^2 + ---\}\{g_m(x)\}$ ---(3)

Which can be simplified by using the fact, that $D = \frac{d}{dx}$

In this case we have to remember the following.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + ---$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + ---$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ---$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ---$$

Worked Examples

1. *Solve* $(D^2 + 3D + 2)y = 1 + 3x + x^2$

➡ The AE is $m^2 + 3m + 2 = 0$

$m = -1, -2$

Therefore, C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{x^2 + 3x + 1}{D^2 + 3D + 2}$$
$$= \frac{1}{2\left(1 + \frac{D^2 + 3D}{2}\right)}(x^2 + 3x + 1)$$
$$= \frac{1}{2}\left(1 + \frac{D^2 + 3D}{2}\right)^{-1}(x^2 + 3x + 1)$$
$$= \frac{1}{2}\left(1 - \frac{D^2 + 3D}{2} + \left(\frac{D^2 + 3D}{2}\right)^2 - + ---\right)(x^2 + 3x + 1)$$
$$= \frac{1}{2}\left(1 - \frac{D^2}{2} - \frac{3D}{2} + \frac{1}{4}(D^4 + 9D^2 + 6D^3) - + ---\right)(x^2 + 3x + 1)$$

$$=\frac{1}{2}\left(x^2+3x+1-\frac{1}{2}(2)-\frac{3}{2}(2x+3)+\frac{1}{4}(0+9(2)+0)\right)$$

$$=\frac{1}{2}\left(x^2+3x+1-1-3x-\frac{9}{2}+\frac{9}{2}\right)$$

$$\text{P.I.}=\frac{1}{2}x^2$$

Hence, the general solution is

$$y=c_1e^{-x}+c_2e^{-2x}+\frac{1}{2}x^2$$

■

2. *Solve* $y''-2y'+y=x^2$

➡ $y''-2y'+y=x^2$

$$(D^2-2D+1)y=x^2$$

The A.E. is $m^2-2m+1=0$

$$(m-1)^2=0$$

$$m=1,1$$

Therefore,

$$\text{C.F.}=(c_1+c_2x)e^x$$

$$\text{P.I.}=\frac{x^2}{D^2-2D+1}=\frac{x^2}{1+(D^2-2D)}=\left[1+(D^2-2D)\right]^{-1}x^2$$

$$=\left[1-(D^2-2D)+(D^2-2D)^2+---\right]x^2$$

$$=\left[1-D^2+2D+D^4+4D^2-4D^3+---\right]x^2$$

$$=x^2-2+2(2x)+0+4(2)-4(0)$$

$$\text{P.I.}=x^2+4x+6$$

Hence, the general solution is

$$y=(c_1+c_2x)e^x+x^2+4x+6$$

■

3. *Solve* $y''+3y'+2y=12x^2$

➡ The given equation can be written as

$$(D^2+3D+2)y=12x^2$$

The A.E. is $m^2 + 3m + 2 = 0$

$$m^2 + 2m + m + 2 = 0$$

$$m(m+2) + 1(m+2) = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

Therefore, $\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{12x^2}{D^2 + 3D + 2} = \frac{12x^2}{2\left(1 + \frac{D^2 + 3D}{2}\right)}$$

$$= \left(1 + \frac{D^2 + 3D}{2}\right)^{-1} 6x^2$$

$$= \left\{1 - \frac{D^2 + 3D}{2} + \left(\frac{D^2 + 3D}{2}\right)^2 + - - -\right\} 6x^2$$

$$= \left(1 - \frac{1}{2}D^2 - \frac{3}{2}D + \frac{1}{4}\left(D^4 + 9D^2 + 6D^3\right) + - - -\right) 6x^2$$

$$= \left\{6x^2 - \frac{1}{2}(12) - \frac{3}{2}(12x) + \frac{1}{4}[0 + 9(12) + 0]\right\}$$

$$= \left\{6x^2 - 6 - 18x + 27\right\}$$

$$\text{P.I.} = 6x^2 - 18x + 21$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + 6x^2 - 18x + 21$$ ■

4. *Solve* $y'' - 4y = x^2$

➡ The given equation can be written as

$$\left(D^2 - 4\right) y = x^2$$

The A.E. is $m^2 - 4 = 0$

$$m = \pm 2$$

Therefore, $\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{x^2}{D^2-4} = \frac{x^2}{-4\left[1-\left(D^2/4\right)\right]} = -\frac{1}{4}\left[1-\frac{D^2}{4}\right]^{-1} x^2$$

$$= -\frac{1}{4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+---\right]x^2$$

$$= -\frac{1}{4}\left[x^2+\frac{1}{4}(2)\right] = -\frac{1}{4}x^2-\frac{1}{8}$$

Hence, the general solution is

$$y = c_1e^{2x}+c_2e^{-2x}-\frac{1}{4}x^2-\frac{1}{8}$$ ■

5. *Solve* $\left(D^2+D-2\right)y = 2\left(1+x-x^2\right)$

➡ The A.E. is $m^2+m-2=0$

$$(m-1)(m+2)=0$$

$$\therefore m = 1, -2$$

Therefore, C.F. $= c_1e^x+c_2e^{-2x}$

$$\text{P.I.} = \frac{2\left(1+x+x^2\right)}{D^2+D-2} = \frac{2\left(1+x-x^2\right)}{-2\left[1-\left(D^2+D\right)/2\right]}$$

$$= -\left(1-\frac{D^2+D}{2}\right)^{-1}\left(1+x-x^2\right)$$

$$= -\left[1+\frac{D^2+D}{2}+\left(\frac{D^2+D}{2}\right)^2+----\right]\left(1+x-x^2\right)$$

$$= -\left[1+\frac{1}{2}D^2+\frac{1}{2}D+\frac{1}{4}\left(D^4+D^2+2D^3\right)+---\right]\left(1+x-x^2\right)$$

$$= -\left[1+x-x^2+\frac{1}{2}(-2)+\frac{1}{2}(1-2x)+\frac{1}{4}(-2)\right]$$

$$= -\left[1+x-x^2-1+\frac{1}{2}-x-\frac{1}{2}\right] = x^2$$

Hence, the general solution is

$$y = c_1e^x+c_2e^{-2x}+x^2$$ ■

6. *Solve* $(D^3+8)y = x^4+2x+1$

➡ The A.E. is $m^3+8=0$

$$m^3+2^3=0$$
$$(m+2)(m^2-2m+2^2)=0$$
$$m=-2 \;\; or \;\; m^2-2m+4=0$$
$$m=\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}=\frac{-2\pm\sqrt{-12}}{2}=-\frac{+2\pm2\sqrt{3}i}{2}$$
$$m=+1\pm\sqrt{3}i$$
$$m=+2,-1\pm\sqrt{3}i$$

Therefore, C.F. = $c_1e^{-2x}+e^{-x}\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)$

$$\text{P.I.}=\frac{x^4+2x+1}{D^3+8}=\frac{x^4+2x+1}{D^3+8}$$
$$=\frac{x^4+2x+1}{8\left(1+\frac{D^3}{8}\right)}=\frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}\left(x^4+2x+1\right)$$
$$=\frac{1}{8}\left[1-\frac{D^3}{8}+\left(\frac{D^3}{8}\right)^2----\right]\left(x^4+2x+1\right)$$
$$=\frac{1}{8}\left[x^4+2x+1-\frac{1}{8}(24x)\right]=\frac{1}{8}\left[x^4+2x+1-3x\right]$$

Thus, $\quad \text{P.I.}=\frac{1}{8}\left(x^4-x+1\right)$

Hence, the general solution is

$$y=c_1e^{-2x}+e^{-x}\left(c_2\cos\sqrt{3x}+c_3\sin\sqrt{3x}\right)+\frac{1}{8}\left(x^4-x+1\right) \qquad ■$$

7. *Solve* $\frac{d^3y}{dx^3}-7\frac{d^2y}{dx^2}+6y=1-x+x^2$

➡ The AE is $m^3-7m^2+6=0$

Here, $m=1$ is one of the roots

$m=1$				
	1	-7	0	6
	0	1	-6	-6
	1	-6	-6	**0**

$m^2-6m-6=0$

$$m=\frac{6\pm\sqrt{36+24}}{2}=\frac{6\pm2\sqrt{15}}{2}=3\pm\sqrt{15}$$

$m=1,\ 3+\sqrt{15},\ 3-\sqrt{15}$

Therefore, C.F. $=c_1e^x+c_2e^{(3+\sqrt{15})x}+c_3e^{(3-\sqrt{15})x}$

$$\text{P.I.}=\frac{1-x+x^2}{D^3-7D+6}=\frac{1-x+x^2}{6\left[1+\left(D^3-7D\right)/6\right]}$$

$$=\frac{1}{6}\left(1+\frac{D^3-7D}{6}\right)^{-1}\left(1-x+x^2\right)$$

$$=\frac{1}{6}\left[1-\frac{D^3-7D}{6}+\frac{1}{36}\left(D^6+49D^2-14D^4\right)+--\right]\left(1-x+x^2\right)$$

$$=\frac{1}{6}\left[1-x+x^2-\frac{1}{6}\left(0-7(-1+2x)\right)+\frac{1}{36}\left(0+49(2)-0\right)\right]$$

$$=\frac{1}{6}\left[1-x+x^2+\frac{7}{6}(2x-1)+\frac{49}{18}\right]$$

$$=\frac{1}{108}\left[18-18x+18x^2+42x-21+49\right]$$

$$=\frac{1}{108}\left(18x^2+24x+46\right)=\frac{1}{54}\left(9x^2+12x+23\right)$$

Hence, the general solution is

$$y=c_1e^x+c_2e^{(3+\sqrt{15})x}+c_3e^{(3-\sqrt{15})x}+\frac{1}{54}\left(9x^2+12x+23\right)$$ ■

8. *Solve* $\left(D^2+D+1\right)y=1+x+x^2$

➡ The A.E. is $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Therefore, C.F. $= e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}\left(1 + x + x^2\right)$$

$$= \left[1 + \left(D^2 + D\right)\right]^{-1}\left(1 + x + x^2\right)$$

$$= \left[1 - \left(D^2 + D\right) + \left(D^2 + D\right)^2 - - - -\right]\left(1 + x + x^2\right)$$

$$= \left[1 - \left(D^2 + D\right) + \left(D^4 + D^2 + 2D^3\right) - - - -\right]\left(1 + x + x^2\right)$$

$$= \left(1 + x + x^2\right) - \left(2 + 1 + 2x\right) + 2 = x^2 - x$$

Thus, G.S. = C.F. + P.I.

Hence, the general solution is

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) + x^2 - x \quad \blacksquare$$

Exercises

1) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x$

2) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x$

3) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = x + k$

4) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

5) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + x + e^{2x}$

6) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$

7) $\left(D^2 - 2D + 1\right)y = x^3 - 6x^2$

8) $\left(D^2 + 4\right)y = x^2 - x$

9) $\left(D^2 + 4D + 3\right)y = x^3$

10) $(2D^2+3D+4)y = x^2 - 2x$

11) $(D^2-4D+4)y = x^2$

12) $\dfrac{d^2y}{dx^2}+4y = x^2+e^x+\sin 3x$

13) $y''+4y = x^2+\sin 2x$

14) $y''+5y'+6y = \cos^2 2x + x$

15) $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx} = 1+x^2$

16) $(D^3-2D^2-3D)y = 3x^2+\sin x$

17) $(D^3+3D^2-D+3)y = x^2$

18) $(D^3-4D^2+5D-2)y = x$

19) $y'''+2y''+y' = x^2+x+e^{2x}$

20) $(D^3-8)y = x^3+x+e^{2x}$

21) $(D^3-D^2)y = x^2-3x+1$

22) $(D^3+1)y = x^3$

Answers

1) $y = c_1e^x+c_2e^{2x}+\dfrac{2x+3}{4}$

2) $y = c_1e^{3x}+c_2e^{-2x}-\dfrac{1}{36}(6x-1)$

3) $y = c_1e^{2x}+c_2e^{-3x}-\dfrac{1}{36}(6x-6k+1)$

4) $y = c_1e^{-2x}+c_2e^{x}-\dfrac{1}{4}(2x+1)-\dfrac{1}{10}(3\sin x+\cos x)$

5) $y = c_1+(c_2+c_3x)e^{-x}+\dfrac{1}{18}e^{2x}+\dfrac{1}{6}(2x^3-9x^2+24x)$

6) $y = (c_1+c_2x)e^{2x}+\dfrac{1}{8}(2x^2+4x+3)$

7) $y = (c_1+c_2x)e^{x}+x^3-6x-12$

8) $y = c_1\cos 2x+c_2\sin 2x+\dfrac{1}{8}(2x^2-2x-1)$

9) $y = c_1e^{-x}+c_2e^{-3x}+\dfrac{1}{3}x^3-\dfrac{4}{3}x^2+\dfrac{26}{9}x-\dfrac{80}{27}$

10) $y = e^{\left(-\frac{3}{4}\right)x}\left[c_1\cos\left(\dfrac{\sqrt{23}}{4}x\right)+c_2\sin\left(\dfrac{\sqrt{23}}{4}x\right)\right]+\dfrac{1}{32}(8x^2-28x+13)$

11) $y=(c_1+c_2x)e^{2x}+\frac{1}{8}(2x^2+4x+3)$

12) $y=c_1\cos 2x+c_2\sin 2x+\frac{1}{5}e^{x}-\frac{1}{5}\sin 3x+\frac{1}{4}x^2-\frac{1}{8}$

13) $y=c_1\cos 2x+c_2\sin 2x+\frac{1}{8}(2x^2-2x\cos 2x-1)$

14) $y=c_1e^{-2x}+c_2e^{-3x}+\frac{1}{6}x-\frac{1}{100}(\cos 4x-2\sin 4x)-\frac{1}{18}$

15) $y=c_1+c_2e^{3x}+c_3e^{-2x}-\frac{1}{108}(6x^3-3x^2+25x)$

16) $y=c_1+c_2e^{-x}+c_3e^{3x}+\frac{x}{9}(6x-14)+\frac{1}{10}(2\cos x+\sin x)$

17) $y=c_1e^{x}+c_2e^{-x}+c_3e^{3x}+\frac{1}{27}(9x^2+6x+20)$

18) $y=(c_1+c_2x)e^{x}+c_3e^{2x}-\frac{x}{2}-\frac{5}{4}$

19) $y=c_1+(c_2+c_3x)e^{-x}+\frac{x}{6}(2x^2-9x+24)+\frac{e^{2x}}{18}$

20) $y=c_1e^{2x}+e^{-x}(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x)$
$\quad +\frac{1}{12}xe^{2x}-\frac{1}{32}(4x^3+4x+3)$

21) $y=c_1+c_2x+c_3e^{x}-\frac{x^3}{12}(x-3)$

22) $y=c_1e^{-x}+e^{\frac{x}{2}}\left\{c_2\cos\left(\frac{\sqrt{3}}{2}x\right)+c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}+x^3-6$

Particular integral when $f(x)=e^{ax}v$ ***where*** v ***is any function of*** x

Let u be any function of x

$$D\left(e^{ax}u\right)=e^{ax}Du+ae^{ax}u=e^{ax}(D+a)u$$

$$D^2\left(e^{ax}u\right)=D\left\{e^{ax}(Du)+ae^{ax}u\right\}$$

$$=e^{ax}D^2u+ae^{ax}Du+ae^{ax}Du+a^2e^{ax}u$$

$$=e^{ax}\left(D^2+2aD+a^2\right)u$$

$$D^2\left(e^{ax}u\right)=e^{ax}(D+a)^2u$$

Similarly, $D^3\left(e^{ax}u\right)=e^{ax}(D+a)^3u$

Therefore, $F(D)\left(e^{ax}u\right)=e^{ax}F(D+a)u$ ---(1)

If $F(D+a)u=v$, then $u=\dfrac{1}{F(D+a)}v$

Using this in equation (1), we get

$$F(D)\left[e^{ax}\frac{1}{F(D+a)}v\right]=e^{ax}v$$

Operating $\dfrac{1}{F(D)}$ on both sides, we get

$$e^{ax}\frac{1}{F(D+a)}v=\frac{1}{F(D)}e^{ax}v$$

$$\text{or } \frac{1}{F(D)}\left(e^{ax}v\right)=e^{ax}\left\{\frac{1}{F(D+a)}v\right\}$$

In the next step we will use the methods which are discussed earlier.

Worked Examples

1. *Solve* $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-4y=xe^{-2x}$

➡ The AE is $m^2+3m-4=0$

$$(m+4)(m-1)=0$$

$$m=-4,1$$

Therefore, C.F. $=c_1e^{-4x}+c_2e^{x}$

$$\text{P.I.}=\frac{xe^{-2x}}{D^2+3D-4}=e^{-2x}\left\{\frac{x}{(D-2)^2+3(D-2)-4}\right\}$$

$$=e^{-2x}\left\{\frac{x}{D^2-4D+4+3D-6-4}\right\}=e^{-2x}\left\{\frac{x}{D^2-D-6}\right\}$$

$$=e^{-2x}\left\{\frac{x}{-6\left[1-\left(D^2-D\right)/6\right]}\right\}=-\frac{1}{6}e^{-2x}\left[1-\frac{D^2-D}{6}\right]^{-1}x$$

$$=-\frac{1}{6}e^{-2x}\left[1+\frac{D^2-D}{6}+---\right]x$$

$$=-\frac{1}{6}e^{-2x}\left[x+\frac{1}{6}(0-1)\right]=-\frac{1}{36}(6x-1)e^{-2x}$$

Hence, the general solution is

$$y=c_1e^{-4x}+c_2e^{x}-\frac{1}{36}(6x-1)e^{-2x}$$ ■

2. *Solve* $\left(D^2-2D+1\right)y=xe^x$

➡ The A.E is $m^2-2m+1=0$

$m=1,1$

$\text{C.F.}=(c_1+c_2x)e^x$

$$\text{P.I.}=\frac{xe^x}{D^2-2D+1}=e^x\cdot\frac{x}{(D+1)^2-2(D+1)+1}$$

$$=e^x\cdot\frac{x}{D^2+2D+1-2D-2+1}=e^x\cdot\frac{x}{D^2}$$

$$=e^x\cdot\frac{1}{D}\int xdx=e^x\cdot\frac{1}{D}\frac{x^2}{2}=e^x\int\frac{x^2}{2}dx$$

$$\text{P.I.}=\frac{1}{6}x^3e^x$$

Hence, the general solution is

$$y=(c_1+c_2x)e^x+\frac{1}{6}x^3e^x$$ ■

3. *Solve* $\left(D^2-4\right)y=8xe^{2x}$

➡ The A.E is $m^2-4=0$

$m=\pm 2$

Therefore, C.F $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{8xe^{2x}}{D^2 - 4} = 8e^{2x} \cdot \frac{x}{(D+2)^2 - 4}$$

$$= 8e^{2x} \cdot \frac{x}{D^2 + 4D + 4 - 4}$$

$$= 8e^{2x} \cdot \frac{x}{4D\left(1 + \frac{D}{4}\right)} = 8e^{2x} \cdot \frac{1}{4D}\left(1 + \frac{D}{4}\right)^{-1} x$$

$$= 2e^{x} \cdot \frac{1}{D}\left[1 - \frac{D}{4} + \left(\frac{D}{4}\right)^2 - + - - -\right] x$$

$$= 2e^{x} \frac{1}{D}\left(x - \frac{1}{4}\right) = 2e^{x} \cdot \int\left(x - \frac{1}{4}\right) dx$$

$$= 2e^{x}\left(\frac{x^2}{2} - \frac{x}{4}\right)$$

$$\text{P.I.} = \left(x^2 - \frac{1}{2}\right) e^{x}$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + \left(x^2 - \frac{1}{2}\right) e^{x}$$ ■

4. *Solve* $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \sin 2x$

➡ The AE is $m^2 + 5m + 6 = 0$

$m = -2, -3$

Therefore, C.F $= c_1 e^{-2x} + c_2 e^{-3x}$

$$\text{P.I.} = \frac{e^{-2x} \sin 2x}{D^2 + 5D + 6} = e^{-2x}\left\{\frac{\sin 2x}{(D-2)^2 + 5(D-2) + 6}\right\}$$

$$= e^{-2x}\left\{\frac{\sin 2x}{D^2 - 4D + 4 + 5D - 10 + 6}\right\} = e^{-2x}\left\{\frac{\sin 2x}{D^2 + D}\right\}$$

$$= e^{-2x}\left\{\frac{\sin 2x}{-2^2+D}\right\} = e^{-2x}\left\{\frac{(D+4)}{(D-4)(D+4)}\sin 2x\right\}$$

$$= e^{-2x}\left\{\frac{(D+4)}{D^2-4^2}\sin 2x\right\} = e^{-2x}\left\{\frac{D+4}{-2^2-16}\right\}\sin 2x$$

$$= -\frac{1}{20}e^{-2x}(D+4)\sin 2x$$

$$= -\frac{1}{20}(2\cos 2x + 4\sin 2x) = -\frac{1}{10}(\cos 2x + 2\sin 2x)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} - \frac{1}{10}(\cos 2x + 2\sin 2x) \qquad \blacksquare$$

5. *Solve* $y'' - y = (1+x^2)e^x + x\sin x$

➡ The AE is $m^2 - 1 = 0$

$$m = 1, -1$$

Therefore, $\text{C.F.} = c_1 e^x + c_2 e^{-x}$

$$\text{P.I.} = \frac{(1+x^2)e^x + x\sin x}{D^2-1} = \frac{(1+x^2)e^x}{D^2-1} + \frac{x\sin x}{D^2-1} = \text{P.I}_1 + \text{P.I}_2$$

$$\text{P.I}_1 = \frac{(1+x^2)e^x}{D^2-1} = e^x\left\{\frac{1+x^2}{(D+1)^2-1}\right\}$$

$$= e^x\left\{\frac{1}{D^2+2D+1-1}(1+x^2)\right\}$$

$$= e^x\left\{\frac{1}{2D(1+(D/2))}(1+x^2)\right\}$$

$$= \frac{1}{2}e^x\left\{\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)\right\}$$

$$= \frac{1}{2}e^x\left\{\frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4}+---\right)(1+x^2)\right\}$$

$$=\frac{1}{2}e^{x}\left\{\frac{1}{D}\left(1+x^{2}-\frac{1}{2}(2x)+\frac{1}{4}(2)\right)\right\}$$

$$=\frac{1}{2}e^{x}\left\{\frac{1}{D}\left(1+x^{2}-x+\frac{1}{2}\right)\right\}$$

$$=\frac{1}{2}e^{x}\left\{\int\left(x^{2}-x+\frac{3}{2}\right)dx\right\}=\frac{1}{2}e^{x}\left[\frac{x^{3}}{3}-\frac{x^{2}}{2}+\frac{3x}{2}\right]$$

$$=\frac{1}{12}e^{x}\left(2x^{3}-3x^{2}+9x\right)$$

$$\text{P.I}_2=\frac{x\sin x}{D^{2}-1}=\text{I.P of }\frac{xe^{ix}}{D^{2}-1}$$

$$=\text{I.P. of }\left\{e^{ix}\left[\frac{x}{(D+i)^{2}-1}\right]\right\}$$

$$=\text{I.P. of }\left\{e^{ix}\left[\frac{x}{D^{2}+2iD+i^{2}-1}\right]\right\}$$

$$=\text{I.P. of }e^{ix}\left\{\frac{x}{-2\left[1-\left(D^{2}+2iD\right)/2\right]}\right\}$$

$$=\text{I.P. of }e^{ix}\left\{-\frac{1}{2}\left[1-\frac{D^{2}+2iD}{2}\right]^{-1}x\right\}$$

$$=\text{I.P. of }e^{ix}\left\{-\frac{1}{2}\left(1+\frac{D^{2}+2iD}{2}+---\right)x\right\}$$

$$=\text{I.P. of }e^{ix}\left\{-\frac{1}{2}\left(x+\frac{1}{2}\left(0+2i(1)\right)+---\right)\right\}$$

$$=\text{I.P. of }e^{ix}\left\{-\frac{1}{2}(x+i)\right\}=-\frac{1}{2}\text{I.P of }e^{ix}(x+i)$$

$$=-\frac{1}{2}\text{I.P of}(\cos x+i\sin x)(x+i)$$

$$=-\frac{1}{2}\text{I.P.of}\left(x\cos x+ix\sin x+i\cos x-\sin x\right)$$

$$\text{P.I}_2=-\frac{1}{2}\left(x\sin x+\cos x\right)$$

$$\text{P.I.}=\frac{1}{12}e^{x}\left(2x^{3}-3x^{2}+9x\right)-\frac{1}{2}\left(x\sin x+\cos x\right)$$

Hence, the general solution is

$$y=c_1e^{x}+c_2e^{-x}+\frac{1}{12}e^{x}\left(2x^{3}-3x^{2}+9x\right)-\frac{1}{2}\left(x\sin x+\cos x\right) \blacksquare$$

6. *Solve* $\left(D^2-1\right)y=x^2\cos x$.

➡ The AE is $m^2-1=0$

$m=1,-1$

Therefore, $\text{C.F}=c_1e^{x}+c_2e^{-x}$

$$\text{P.I.}=\frac{x^2\cos x}{D^2-1}=\text{R.P. of }\frac{x^2e^{ix}}{D^2-1}$$

$$=\text{R.P. of }e^{ix}\left\{\frac{x^2}{(D+i)^2-1}\right\}$$

$$=\text{R.P. of }e^{ix}\left\{\frac{x^2}{D^2+2iD+i^2-1}\right\}$$

$$=\text{R.P. of }e^{ix}\left\{\frac{x^2}{-2\left(1-\left(D^2+2iD\right)/2\right)}\right\}$$

$$=-\frac{1}{2}\text{R.P.of }e^{ix}\left\{\left(1-\frac{D^2+2iD}{2}\right)^{-1}x^2\right\}$$

$$=-\frac{1}{2}\text{R.P. of }e^{ix}\left\{1+\frac{D^2+2iD}{2}+\left(\frac{D^2+2iD}{2}\right)^2+---\right\}x^2$$

$$=-\frac{1}{2}\text{R.P.of }e^{ix}\left\{1+\frac{1}{2}\left(D^2+2iD\right)+\frac{1}{4}\left(D^4-4D^2+4iD^3\right)+---\right\}x^2$$

$$= -\frac{1}{2}\text{R.P.of } e^{ix}\left\{x^2 + \frac{1}{2}(2+4ix) + \frac{1}{4}(0-4(2)+0)\right\}$$

$$= -\frac{1}{2}\text{ R.P.of } e^{ix}\{(x^2+1+2ix)-2\}$$

$$= -\frac{1}{2}\text{R.P. of } e^{ix}\{(x^2-1)+i(2x)\}$$

$$= -\frac{1}{2}\text{R.P. of }(\cos x + i\sin x)\{(x^2-1)+i(2x)\}$$

$$\text{P.I.} = -\frac{1}{2}\{(x^2-1)\cos x - 2x\sin x\}$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-x} - \frac{1}{2}\{(x^2-1)\cos x - 2x\sin x\}$$ ■

7. *Solve* $(D^2 - 2D + 1)y = xe^x \sin x$

➡ The A.E is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

Therefore,

$$\text{C.F} = (c_1 + c_2x)e^x$$

$$\text{P.I.} = \frac{xe^x \sin x}{D^2 - 2D + 1} = e^x \cdot \frac{x\sin x}{D^2 - 2D + 1}$$

$$= e^x \cdot \frac{x\sin x}{(D+1)^2 - 2(D+1) + 1}$$

$$= e^x \cdot \frac{x\sin x}{D^2 + 2D + 1 - 2D - 2 + 1}$$

$$= e^x \cdot \frac{x\sin x}{D^2} = e^x \cdot \frac{1}{D}\int x\sin x dx$$

$$= e^x \cdot \frac{1}{D}[x(-\cos x) - (1)(-\sin x)]$$

$$= e^x \cdot \frac{1}{D}[-x\cos x + \sin x]$$

$$= e^x \int (\sin x - x\cos x)dx$$

$$= e^x\{-\cos x - [x\sin x - (1)(-\cos x)]\}$$

$$= e^x\{-\cos x - x\sin x - \cos x\}$$

$$= e^x\{-x\sin x - 2\cos x\}$$

$$\text{P.I.} = -e^x(x\sin x + 2\cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^x - e^x(x\sin x + 2\cos x)$$

8. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8x^2 e^{2x}\sin 2x$

➡ The AE is $m^2 - 4m + 4 = 0$

$m = 2, 2$

Therefore, C.F. $= (c_1 + c_2 x)e^{2x}$

$$\text{P.I.} = \frac{8x^2 e^{2x}\sin 2x}{D^2 - 4D + 4} = \frac{8x^2 e^{2x}\sin 2x}{(D-2)^2}$$

$$= 8e^{2x}\left\{\frac{x^2\sin 2x}{(D+2-2)^2}\right\} = 8e^{2x}\left\{\frac{x^2\sin 2x}{D^2}\right\}$$

$$= 8e^{2x}\frac{1}{D}\int x^2\sin 2x dx$$

$$= 8e^{2x}\frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right) - \int 2x\left(-\frac{\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \int x\cos 2x dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + x\left(\frac{\sin 2x}{2}\right) - \int 1\cdot\frac{\sin 2x}{2}dx\right]$$

$$= e^{2x}\int\left[-4x^2\cos 2x + 4x\sin 2x + 2\cos 2x\right]dx$$

$$= e^{2x}\left[-4\left\{x^2\left(\frac{\sin 2x}{2}\right)-\int 2x\frac{\sin 2x}{2}dx\right\}\right.$$

$$\left.+4\int x\sin 2x\,dx+2\frac{\sin 2x}{2}\right]$$

$$= e^{2x}\left[-2x^2\sin 2x+4\int x\sin 2x\,dx+4\int x\sin 2x\,dx+\sin 2x\right]$$

$$= e^{2x}\left[-2x^2\sin 2x+8\int x\sin 2x\,dx+\sin 2x\right]$$

$$= e^{2x}\left\{-2x^2\sin 2x+8\left[x\left(-\frac{\cos 2x}{2}\right)-\int 1\left(-\frac{\cos 2x}{2}\right)dx\right]\right.$$

$$\left.+\sin 2x\right\}$$

$$= e^{2x}\left\{-2x^2\sin 2x-4x\cos 2x+4\left(\frac{\sin 2x}{2}\right)+\sin 2x\right\}$$

$$= e^{2x}\left[-2x^2\sin 2x-4x\cos 2x+3\sin 2x\right]$$

$$\text{P.I.} = e^{2x}\left[\left(3-2x^2\right)\sin 2x-4x\cos 2x\right]$$

Hence, the general solution is

$$y=(c_1+c_2x)e^{2x}+e^{2x}\left[\left(3-2x^2\right)\sin 2x-4x\cos 2x\right]$$ ■

Exercises

1) $y''+y=4x\sin x$
2) $y''+y=e^x\sin 2x$
3) $y''+4y=x\sin^2 x$
4) $y''-2y'+y=e^x\sin x$
5) $y''+2y'+3y=e^{-x}\cos x$
6) $y''-4y'+4y=e^{2x}\cos^2 x$
7) $y''-5y'+6y=4x^2e^x$
8) $y''-9y'+20y=x^2e^{3x}$
9) $\left(D^3-3D+2\right)y=x^2e^x$
10) $\left(D^3+3D^2+3D+1\right)y=xe^{-x}$
11) $\left(D^3-6D^2+12D-8\right)y=x^2e^{2x}$
12) $\left(D^4-1\right)y=e^x\cos x$
13) $\left(D^2+1\right)y=e^x\cos x$

Answers

1) $y = c_1 \cos x + c_2 \sin x + x \sin x - x^2 \cos x$

2) $y = c_1 \cos x + c_2 \sin x - \frac{1}{10} e^x (\sin 2x + 2\cos 2x)$

3) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{8} - \frac{x}{32} \cos 2x - \frac{x^2}{16} \sin 2x$

4) $y = (c_1 + c_2 x) e^x - e^x \sin x$

5) $y = e^{-x} \left(c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x \right) + e^{-x} \cos x$

6) $y = (c_1 + c_2 x) e^{2x} + \frac{1}{8} e^{2x} \left(2x^2 - \cos 2x \right)$

7) $y = c_1 e^{2x} + c_3 e^{3x} + \left(2x^2 + 6x + 7 \right) e^x$

8) $y = c_1 e^{4x} + c_2 e^{5x} + \frac{1}{4} \left(2x^2 + 6x + 7 \right) e^{3x}$

9) $y = (c_1 + c_2 x) e^x + c_3 e^{-2x} + \left(\frac{x^4}{36} - \frac{x^3}{27} + \frac{x^2}{27} \right) e^x$

10) $y = \left(c_1 + c_2 x + c_3 x^2 \right) e^{-x} + \frac{1}{24} x^4 e^{-x}$

11) $y = \left(c_1 + c_2 x + c_3 x^2 \right) e^{2x} + \frac{1}{60} x^5 e^{2x}$

12) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} e^x \cos x$

13) $y = c_1 \cos x + c_2 \sin x + \frac{e^x}{5} (\cos x + 2 \sin x)$