Inverse Differential Operator and Particular Integral

Consider the differential equation

$$F(D)y = f(x) \quad \text{---(1)}$$

Operating $1/F(D)$ on both the sides, we get

$$\frac{1}{F(D)}\{F(D)y\} = \frac{1}{F(D)} f(x) \quad \text{---(2)}$$

Here $1/F(D)$ is called inverse differential operator and (2) becomes,

$$y = \frac{1}{F(D)} f(x)$$

This is the solution of the differential equation (1) called the particular integral.

Basic Method of Finding Particular Integral

Let us consider the 1st order linear differential equation with constant coefficients of the form

$$(D-a)y = f(x) \quad \text{---(1)}$$

A.E. is $\quad m - a = 0$

$\quad m = a$

$\therefore$ C.F. $= c_1 e^{ax}$

Given equation is $(D-a)y = f(x)$

$$\frac{dy}{dx} - ay = f(x)$$

Which is 1st order linear equation with

$$P = -a \quad \text{and} \quad Q = f(x)$$

$$\text{I.F.} = e^{\int P dx} = e^{-\int a dx} = e^{-ax}$$

Therefore, general solution of the given equation is

$$y(I.F.) = \int (I.F.) Q dx + c$$

$$ye^{-ax} = \int e^{-ax} f(x) dx + c$$

$$\therefore y = \frac{1}{e^{-ax}}\left\{\int e^{-ax} f(x) dx + c\right\}$$

i.e., $y = e^{ax}\left\{\int e^{-ax} f(x)dx + c\right\}$

i.e., $y = e^{ax}\int e^{-ax} f(x)dx + ce^{ax}$ ---(2)

We know that the general solution for non-homogeneous equation is

G.S. = C.F. + P.I.

Hence, from equation (2), we have

P.I. $= e^{ax}\int e^{-ax} f(x)dx$

i.e., P.I. $= \dfrac{1}{D-a} f(x) = e^{ax}\int e^{-ax} f(x)dx$

Note If $a = 0 \Rightarrow \dfrac{1}{D} f(x) = \int f(x)dx$

Worked Examples

1. *Solve* $(D^2 - 3D + 2)y = x$

➡ The A.E. is $m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

Therefore,

C.F. $= c_1e^x + c_2e^{2x}$

P.I. $= \dfrac{x}{D^2 - 3D + 2} = \dfrac{x}{(D-1)(D-2)} = \dfrac{1}{D-1}\left\{\dfrac{1}{D-2}(x)\right\}$

$= \dfrac{1}{D-1}\left\{e^{2x}\int e^{-2x} x dx\right\}$

$= \dfrac{1}{D-1}\left\{e^{2x}\left[x\dfrac{e^{-2x}}{-2} - \int 1\dfrac{e^{-2x}}{-2}dx\right]\right\}$

$= \dfrac{1}{D-1}\left\{e^{2x}\left[-\dfrac{x}{2}e^{-2x} + \dfrac{1}{2}\dfrac{e^{-2x}}{(-2)}\right]\right\}$

$$= \frac{1}{D-1}\left\{-\frac{x}{2}-\frac{1}{4}\right\} = -\frac{1}{4}\frac{1}{D-1}\{(2x+1)\}$$

$$= -\frac{1}{4}e^{x}\int e^{-x}(2x+1)dx$$

$$= -\frac{1}{4}e^{x}\left\{(2x+1)\frac{e^{-x}}{-1} - \int 2\frac{e^{-x}}{-1}dx\right\}$$

$$= -\frac{1}{4}e^{x}\left\{-(2x+1)e^{-x} + 2\frac{e^{-x}}{-1}\right\} = \frac{2x+1+2}{4} = \frac{2x+3}{4}$$

Thus, G.S. = C.F. + P.I.

Hence, $y = c_1e^{x} + c_2e^{2x} + \frac{2x+3}{4}$ ■

2. *Solve* $y'' + 3y' + 2y = \sin(e^{x})$

➡ The A.E. is $m^2 + 3m + 2 = 0$

$$m^2 + m + 2m + 2 = 0$$

$$m(m+1) + 2(m+1) = 0$$

$$(m+2)(m+1) = 0$$

$$m = -2, -1$$

Therefore, C.F. $= c_1e^{-2x} + c_2e^{-x}$

$$\text{P.I.} = \frac{\sin(e^{x})}{(D^2+3D+2)} = \frac{\sin(e^{x})}{(D+2)(D+1)}$$

$$= \frac{1}{D+2}\left\{\frac{1}{D+1}\sin(e^{x})\right\} = \frac{1}{D+2}\left\{e^{-x}\int e^{x}\sin(e^{x})dx\right\}$$

$$= \frac{1}{D+2}\left\{e^{-x}\int \sin t\,dt\right\} = \frac{1}{D+2}\left\{e^{-x}(-\cos t)\right\}$$

$$= \frac{-1}{D+2}\left\{e^{-x}\cos(e^{x})\right\} = -e^{-2x}\int e^{2x}\left(e^{-x}\cos(e^{x})\right)dx$$

$$= -e^{-2x}\int e^{x}\cos(e^{x})dx = -e^{-2x}\int \cos t\,dt = -e^{-2x}\sin t$$

$$\text{P.I.} = -e^{-2x}\sin e^{x}$$

Thus, G.S. = C.F. + P.I.

Hence, $y = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x} \sin e^{x}$ ■

Particular integral when $f(x) = ke^{ax}$

Consider the differential equation

$$F(D)y = f(x)$$

$$F(D)y = ke^{ax}$$

Therefore, P.I. $= \dfrac{1}{F(D)} ke^{ax}$ ---(1)

We have, $F(D)e^{ax} = \left(a_n D^n + a_{n-1} D^{n-1} + --- + a_2 D^2 + a_1 D + a_0\right) e^{ax}$

$$= \left(a_n a^n + a_{n-1} a^{n-1} + --- + a_2 a^2 + a_1 a + a_0\right) e^{ax}$$

$$F(D)e^{ax} = F(a)e^{ax}$$

$$\Rightarrow \frac{1}{F(a)} e^{ax} = \frac{1}{F(D)} e^{ax}$$

$$\Rightarrow \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax} \quad \text{---(2)}$$

Since $De^{ax} = ae^{ax}$, $D^2 e^{ax} = a^2 e^{ax} --- D^n e^{ax} = a^n e^{ax}$

Using equation (2) in (1), we get

$$\text{P.I.} = \frac{1}{F(a)} ke^{ax} \text{ provided } F(a) \neq 0$$

If $F(a) = 0$, then the case fails and we proceed as follows

$F(a) = 0 \Rightarrow a$ is a root of A.E.

$\Rightarrow (D - a)$ is a factor of $F(D)$

$\Rightarrow F(D) = (D - a)\phi(D)$

$$\Rightarrow \text{P.I.} = \frac{1}{F(D)} ke^{ax} = \frac{1}{(D-a)\phi(D)} ke^{ax}$$

$$= \frac{1}{(D-a)\phi(a)} ke^{ax} = \frac{k}{\phi(a)} \frac{1}{D-a} e^{ax}$$

$$= \frac{k}{\phi(a)}\left\{e^{ax}\int e^{-ax}\cdot e^{ax}\,dx\right\} = \frac{k}{\phi(a)}\left\{e^{ax}\cdot x\right\} = x\frac{1}{\phi(a)}ke^{ax}$$

Since $F(D) = (D-a)\phi(D)$

$$F'(D) = (D-a)\phi'(D) + 1\cdot\phi(D)$$

$$F'(a) = 0 + \phi(a) = \phi(a)$$

Thus, P.I.= $x\cdot\frac{1}{F'(a)}ke^{ax}$, provided $F'(a) \neq 0$

If $F'(a) = 0$, then P.I. $= x^2\cdot\frac{1}{F''(a)}ke^{ax}$ provided $F''(a) \neq 0$

If $F''(a) = 0$, then P.I. $= x^3\cdot\frac{1}{F'''(a)}ke^{ax}$ provided $F'''(a) \neq 0$, and so on.

Working Rule

If $f(x) = ke^{ax}$, then P.I. $= \frac{1}{F(D)}ke^{ax} = \frac{1}{F(a)}ke^{ax}$, $F(a) \neq 0$

If $F(a) = 0$, then P.I. $= x\cdot\frac{1}{F'(a)}ke^{ax}$, $F'(a) \neq 0$.

If $F'(a) = 0$ then P.I. $= x^2\cdot\frac{1}{F''(a)}ke^{ax}$, $F''(a) \neq 0$ and so on.

Note If $f(x) = k$, then

$$\text{P.I.} = \frac{k}{F(D)} = \frac{ke^{0\cdot x}}{F(D)} = \frac{ke^{0\cdot x}}{F(0)} = \frac{k}{F(0)}, \text{ provided } F(0) \neq 0$$

Worked Examples

1. *Solve* $y'' - y' - 2y = e^{3x}$

➡ The given equation can be written as

$$(D^2 - D - 2)y = e^{3x}$$

The AE is $m^2 - m - 2 = 0$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

Therefore,

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-x}$$

$$\text{P.I.} = \frac{e^{3x}}{D^2 - D - 2} = \frac{e^{3x}}{4}$$

Thus, G.S. = C.F. + P.I.

Hence, $y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4} e^{3x}$ ■

2. *Solve* $(D^2 - 4D + 3)y = e^x$

➡ The AE is $m^2 - 4m + 3 = 0$

$$(m-1)(m-3) = 0$$

$$m = 1, 3$$

Therefore,

$$\text{C.F.} = c_1 e^{x} + c_2 e^{3x}$$

$$\text{P.I.} = \frac{e^x}{D^2 - 4D + 3} = x \cdot \frac{e^x}{2D - 4} = -\frac{1}{2} x e^x$$

Thus, G.S. = C.F. + P.I.

Hence, $y = c_1 e^x + c_2 e^{3x} - \frac{1}{2} x e^x$ ■

Note If the RHS of the given differential equation is a part of C.F. then we get zero in the denominator when finding P.I.

3. *Solve* $y'' + 5y' + 6y = e^{-2x}$

➡ The given equation can be written as

$$(D^2 + 5D + 6)y = e^{-2x}$$

The AE is $m^2 + 5m + 6 = 0$

$$(m+2)(m+3) = 0$$

$$m = -2, -3$$

Therefore,

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{P.I.} = \frac{e^{-2x}}{D^2 + 5D + 6} = x \cdot \frac{e^{-2x}}{2D + 5} = x e^{-2x}$$

Thus, G.S = C.F + P.I.

Hence, $y = c_1 e^{-2x} + c_2 e^{-3x} + x e^{-2x}$ ■

4. *Solve* $(D^2+D-2)y = 3e^x$

➡ The A.E is $m^2+m-2=0$

$$m^2+2m-m-2=0$$
$$m(m+2)-1(m+2)=0$$
$$(m-1)(m+2)=0$$
$$m=1,\ -2$$

Therefore,

$$\text{C.F} = c_1e^x + c_2e^{-2x}$$

$$\text{P.I} = \frac{3e^x}{D^2+D-2}$$

$$= x\cdot\frac{3e^x}{2D+1} = x\cdot\frac{3e^x}{3}$$

$$\text{P.I} = xe^x$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-2x} + xe^x$$ ■

5. *Solve* $y'' - 6y' + 9y = e^x$

➡ The A.E is $m^2-6m+9=0$

$$(m-3)^2=0$$
$$m=3,\ 3$$

Therefore,

$$\text{C.F} = (c_1+c_2x)e^{3x}$$

$$\text{P.I} = \frac{e^x}{D^2-6D+9}$$

$$= \frac{e^x}{4} = \frac{1}{4}e^x$$

Hence, the general solution is

$$y = (c_1+c_2x)e^{3x} + \frac{1}{4}e^x$$ ■

6. *Solve* $(D^2-13D+12)y = e^{2x}+5e^x$

➡ The A.E is $m^2-13m+12=0$

$$m^2 - 12m - m + 12 = 0$$
$$m(m-12) - 1(m-12) = 0$$
$$(m-1)(m-12) = 0$$
$$m = 1, 12$$

Therefore,

$$\text{C.F} = c_1 e^x + c_2 e^{12x}$$

$$\text{P.I} = \frac{e^{2x} + 5e^x}{D^2 - 13D + 12}$$
$$= \frac{e^{2x}}{D^2 - 13D + 12} + 5\frac{e^x}{D^2 - 13D + 12}$$
$$= \frac{e^{2x}}{4 - 26 + 12} + 5x \cdot \frac{e^x}{2D - 13}$$
$$= \frac{e^{2x}}{-10} + 5x \cdot \frac{e^x}{-11}$$
$$\text{P.I} = -\frac{1}{10}e^{2x} - \frac{5}{11}xe^x$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{12x} - \frac{1}{10}e^{2x} - \frac{5}{11}xe^x$$ ■

7. *Solve* $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x$

➡ The given equation can be written as

$$\left(D^3 - 3D^2 + 4D - 2\right)y = e^x$$

The AE is $m^3 - 3m^2 + 4m - 2 = 0$

$m = 1$ is one of the roots of AE

$m = 1$	1	−3	4	−2
	0	1	−2	2
	1	−2	2	0

$$m^2 - 2m + 2 = 0$$

$$m=\frac{2\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}=1\pm i$$

$$m=1,\ 1\pm i$$

Therefore, C.F. $=c_1e^x+e^x(c_2\cos x+c_3\sin x)$

$$\text{P.I.}=\frac{e^x}{D^3-3D^2+4D-2}=x\cdot\frac{e^x}{3D^2-6D+4}=\frac{xe^x}{1}$$

Thus, G.S. = C.F. + P.I.

Hence, $y=c_1e^x+e^x(c_2\cos x+c_3\sin x)+xe^x$ ■

8. *Solve* $(D^2+7D+12)y=\cosh x$

➡ The AE is $m^2+7m+12=0$

$$m^2+3m+4m+12=0$$

$$m(m+3)+4(m+3)=0$$

$$(m+4)(m+3)=0$$

$$m=-4,-3$$

$$\text{C.F.}=c_1e^{-4x}+c_2e^{-3x}$$

$$\text{P.I.}=\frac{\cosh x}{D^2+7D+12}$$

$$=\frac{1}{D^2+7D+12}\left[\frac{1}{2}\left(e^x+e^{-x}\right)\right]$$

$$=\frac{1}{2}\left[\frac{e^x}{D^2+7D+12}+\frac{e^{-x}}{D^2+7D+12}\right]=\frac{1}{2}\left[\frac{e^x}{20}+\frac{e^{-x}}{6}\right]$$

$$\text{P.I.}=\frac{1}{40}e^x+\frac{1}{12}e^{-x}$$

Thus, G.S. = C.F. + P.I.

Hence, $y=c_1e^{-4x}+c_2e^{-3x}+\frac{1}{40}e^x+\frac{1}{12}e^{-x}$ ■

9. *Solve* $y''-4y=\sinh^2 x$

➡ The given equation can be written as

$$(D^2-4)y=\sinh^2 x$$

The A.E is $m^2 - 4 = 0$

$m = \pm 2$

Therefore, C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4}\sinh^2 x$$

$$= \frac{1}{D^2-4}\left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]^2 = \frac{1}{D^2-4}\left[\frac{1}{4}\left(e^{2x}+e^{-2x}-2\right)\right]$$

$$= \frac{1}{4}\frac{e^{2x}}{\left(D^2-4\right)} + \frac{1}{4}\frac{e^{-2x}}{\left(D^2-4\right)} - \frac{1}{2}\frac{1}{\left(D^2-4\right)}$$

$$= \frac{1}{4}x \cdot \frac{e^{2x}}{(2D)} + \frac{1}{4}x \cdot \frac{e^{-2x}}{(2D)} - \frac{1}{2}\frac{1}{\left(0^2-4\right)}$$

$$= \frac{1}{16}xe^{2x} - \frac{1}{16}xe^{-2x} + \frac{1}{8}$$

$$\text{P.I.} = \frac{1}{16}\left(xe^{2x} - xe^{-2x} + 2\right)$$

Thus, G.S. = C.F. + P.I.

Hence, $y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{16}\left(xe^{2x} - xe^{-2x} + 2\right)$ ■

10. *Solve* $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2\cosh x$

➡ The given equation can be written as

$$\left(D^2 + 4D + 5\right)y = 2\cosh x$$

The AE is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 4(5)(1)}}{2(1)} = -2 \pm i$$

Therefore, C.F. $= e^{-2x}\left(c_1 \cos x + c_2 \sin x\right)$

$$\text{P.I.} = \frac{1}{D^2+4D+5}(2\cosh x) = \frac{1}{D^2+4D+5}\left(e^x + e^{-x}\right)$$

$$= \frac{e^x}{D^2+4D+5} + \frac{e^{-x}}{D^2+4D+5}$$

$$\text{P.I.} = \frac{1}{10}e^{x} + \frac{1}{2}e^{-x}$$

Thus, $\quad \text{G.S.} = \text{C.F.} + \text{P.I.}$

Hence, $\quad y = e^{-2x}\left(c_1 \cos x + c_2 \sin x\right) + \frac{1}{10}e^{x} + \frac{1}{2}e^{-x}$ ■

11. *Solve* $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

➡ The given equation can be written as

$$\left(D^2 - 4\right)y = \cosh(2x-1) + 3^x$$

The AE is $m^2 - 4 = 0$,

$$m = \pm 2$$

Therefore,

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2-4}\left[\cosh(2x-1) + 3^x\right]$$

$$= \frac{1}{D^2-4}\left[\frac{1}{2}\left(e^{2x-1} + e^{-(2x-1)}\right) + 3^x\right]$$

$$= \frac{1}{2}\frac{e^{2x-1}}{D^2-4} + \frac{1}{2}\frac{e^{-2x+1}}{D^2-4} + \frac{3^x}{D^2-4}$$

$$= \frac{1}{2}x \cdot \frac{e^{2x-1}}{2D} + \frac{1}{2}x \cdot \frac{e^{-2x+1}}{2D} + \frac{e^{x\log 3}}{D^2-4}$$

$$= \frac{1}{8}xe^{2x-1} - \frac{1}{8}xe^{-2x+1} + \frac{e^{x\log 3}}{(\log 3)^2 - 4}$$

$$= \frac{1}{4}x\left(\frac{e^{2x-1} - e^{-(2x-1)}}{2}\right) + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{1}{4}x\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Thus, G.S. = C.F. + P.I.

Hence, $\quad y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4}x\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$ ■

12. *Solve* $(D^2+5D+10)y=e^x+4^x$

➡ The AE is $m^2+5m+10=0$

$$m=\frac{-5\pm\sqrt{25-4(1)(10)}}{2(1)}=\frac{-5\pm\sqrt{15}i}{2}=-\frac{5}{2}\pm\frac{\sqrt{15}}{2}i$$

Therefore, C.F. $=e^{-\frac{5}{2}x}\left[c_1\cos\left(\frac{\sqrt{15}}{2}x\right)+c_2\sin\left(\frac{\sqrt{15}}{2}x\right)\right]$

$$\text{P.I.}=\frac{e^x+4^x}{D^2+5D+10}$$

$$=\frac{e^x}{D^2+5D+10}+\frac{4^x}{D^2+5D+10}=\frac{e^x}{16}+\frac{e^{x\log 4}}{D^2+5D+10}$$

$$=\frac{1}{16}e^x+\frac{e^{x\log 4}}{(\log 4)^2+5\log 4+10}$$

$$\text{P.I.}=\frac{1}{16}e^x+\frac{4^x}{(\log 4)^2+5\log 4+10}$$

Thus, G.S. = C.F. + P.I.

Hence,

$$y=e^{-\frac{5}{2}x}\left[c_1\cos\left(\frac{\sqrt{15}}{2}x\right)+c_2\sin\left(\frac{\sqrt{15}}{2}x\right)\right]+\frac{1}{16}e^x+\frac{4^x}{(\log 4)^2+5\log 4+10}$$ ■

Exercises

1) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=e^x$

2) $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=e^x-e^{-2x}$

3) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^{-2x}+2\sinh x$

4) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=(1+e^x)^2$

5) $\frac{d^3y}{dx^3}+y=5e^x+3$

6) $y'''-y''-y'+y=e^x$

7) $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+\frac{dy}{dx}=e^{2x}$

8) $\left(D^2 - a^2\right)y = e^{ax} + e^{nx}, \quad a \neq n$

9) $\dfrac{d^2y}{dx^2} - 2k\dfrac{dy}{dx} + k^2y = e^x, \quad k \neq 1$

10) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 3e^{-2x}$ *given that* $y = 4$ *and* $y' = -7$ *when* $x = 0$

11) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = 2e^{-2t}$

12) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \cosh x$

13) $\dfrac{d^2y}{dx^2} + 4y = \cosh 2x$

14) $y'' - 6y' + 9y = \sinh 3x$

15) $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 10y = \left(1 + e^x\right)^2$

16) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 4 = \sinh(2x + 3)$

17) $\left(D^3 + 2D^2 - D - 2\right)y = 2\cosh x$

18) $\left(D^2 + 5D + 6\right)y = e^{-2x} + e^{-3x}$

19) $\left(D^2 + 5D + 6\right)y = e^x - e^{-2x}$

20) $\left(D^2 - 5D + 6\right)y = \left(e^x + 1\right)^2$

21) $y'' - 2y' + y = 3e^{2x}$

22) $\left(D^3 - D^2 + 4D - 4\right)y = e^x + 2\cosh 3x$

23) $\left(D^3 - D^2 - 2D\right)y = e^{-x}$

24) $\left(D^3 - 3D^2 + 4D - 2\right)y = e^x$

25) $\left(D^3 - 3aD^2 + 3a^2D - a^3\right)y = e^{ax}$

26) $\left(D^4 - D^3 - 3D^2 + 5D - 2\right)y = e^{3x}$

27) $\left(D^4 - 2D^2 + 1\right)y = 4 + e^x$

Answers

1) $y = \left(c_1 + c_2x\right)e^{3x} + \dfrac{1}{4}e^x$

2) $y = c_1e^{-3x} + c_2e^{-2x} + \dfrac{1}{12}e^x - xe^{-2x}$

3) $y = \left(c_1 + c_2x\right)e^x + \dfrac{1}{9}e^{-2x} + \dfrac{1}{2}x^2e^x + \dfrac{1}{4}e^{-x}$

4) $y = c_1e^{2x} + c_2e^{3x} + e^x - xe^{2x} + \dfrac{1}{6}$

5) $y = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] + 3 + \frac{5}{2}e^{x}$

6) $y = (c_1 + c_2 x)e^{x} + c_3 e^{-4x} + \frac{x^2}{4}e^{x}$

7) $y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18}e^{2x}$

8) $y = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{2a}e^{ax} + \frac{1}{a^2 - n^2}e^{nx}$

9) $y = (c_1 + c_2 x)e^{kx} + \frac{1}{k^2 - 2k + 1}e^{x}$

10) $y = e^{-2x}(\cos x + \sin x) + 3e^{-2x}$

11) $x = (c_1 + c_2 t)e^{t} + \frac{2}{9}e^{-2t}$

12) $y = c_1 e^{x} + c_2 e^{2x} - \frac{1}{2}xe^{x} + \frac{1}{12}e^{-x}$

13) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}\cosh 2x$

14) $y = (c_1 + c_2 x)e^{3x} + \frac{x^2}{4}e^{3x} - \frac{1}{36}e^{-3x}$

15) $y = c_1 e^{2x} + c_2 e^{5x} + \frac{1}{10} + \frac{1}{2}e^{x} - \frac{x}{3}e^{2x}$

16) $y = c_1 e^{x} + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{8}\left[\frac{e^{2x+3}}{2} + \frac{e^{-(2x+3)}}{3}\right]$

17) $y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{6}xe^{x} - \frac{1}{2}xe^{-x}$

18) $y = c_1 e^{-2x} + c_2 e^{-3x} + x\left(e^{-2x} - e^{-3x}\right)$

19) $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12}e^{x} - xe^{-2x}$

20) $y = c_1 e^{2x} + c_2 e^{3x} + e^{x} - xe^{2x} + \frac{1}{6}$

21) $y = (c_1 + c_2 x)e^{x} + 3e^{2x}$

22) $y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \dfrac{xe^x}{5} + \dfrac{e^{3x}}{26} - \dfrac{e^{-3x}}{52}$

23) $y = c_1 + c_2 e^{-x} + c_3 e^{2x} + \dfrac{x}{3} e^{-x}$

24) $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + xe^x$

25) $y = (c_1 + c_2 x + c_3 x^2) e^{ax} + \dfrac{x^3}{6} e^{ax}$

26) $y = (c_1 + c_2 x + c_3 x^2) e^x + c_4 e^{-2x} + \dfrac{1}{40} e^{3x}$

27) $y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) e^{-x} + 4 + \dfrac{x^2}{8} e^x$

Particular Integral when $f(x) = k \sin(ax+b)$ ***or*** $k \cos(ax+b)$

We have $D \sin(ax+b) = a \cos(ax+b)$

$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$

i.e., $(D^2) \sin(ax+b) = (-a^2) \sin(ax+b)$

$D^3 \sin(ax+b) = -a^3 \cos(ax+b)$

$D^4 \sin(ax+b) = a^4 \sin(ax+b)$

i.e., $(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$

In general $(D^2)^n \sin(ax+b) = (-a^2)^n \sin(ax+b)$

Therefore, $g(D^2) \sin(ax+b) = g(-a^2) \sin(ax+b)$

$$\Rightarrow \frac{1}{g(D^2)} g(D^2) \sin(ax+b) = \frac{1}{g(D^2)} g(-a^2) \sin(ax+b)$$

$$\Rightarrow \frac{1}{g(-a^2)} \sin(ax+b) = \frac{1}{g(D^2)} \sin(ax+b)$$

$$\Rightarrow \frac{1}{g(D^2)} \sin(ax+b) = \frac{1}{g(-a^2)} \sin(ax+b) \text{ provided } g(-a^2) \neq 0$$

i.e., we replace D^2 by $-a^2$

If $g(-a^2)=0$, then the case fails and we proceed as follows.

We have $\cos(ax+b)+i\sin(ax+b)=e^{i(ax+b)}$

Therefore, $\dfrac{1}{g(D^2)}\sin(ax+b)=\text{I.P. of } \dfrac{e^{i(ax+b)}}{g(D^2)}$

$$=\text{I.P. of } x\cdot\frac{e^{i(ax+b)}}{\frac{d}{dD}\left[g(D^2)\right]}$$

$$=x\cdot\frac{\sin(ax+b)}{g'(D^2)}$$

$\therefore \dfrac{1}{g(D^2)}\sin(ax+b)=x\dfrac{\sin(ax+b)}{g'(-a^2)}$, provided $g'(-a^2)\neq 0$

If $g'(a^2)=0$, then

$$\frac{1}{g(D^2)}\sin(ax+b)=x^2\cdot\frac{\sin(ax+b)}{g''(-a^2)}, \text{ provided } g''(-a^2)\neq 0$$

Similarly,

$\dfrac{1}{g(D^2)}\cos(ax+b)=\dfrac{1}{g(-a^2)}\cos(ax+b)$, provided $g(-a^2)\neq 0$.

If $g(-a^2)=0$, then

$\dfrac{1}{g(D^2)}\cos(ax+b)=x\cdot\dfrac{1}{g'(-a^2)}\cos(ax+b)$, provided $g'(-a^2)\neq 0$.

If $g'(-a^2)=0$, then

$$\frac{1}{g(D^2)}\cos(ax+b)=x^2\frac{1}{g''(-a^2)}\cos(ax+b)$$

where $g'(-a^2)$ and $g''(-a^2)$ denotes the derivatives of $g(D^2)$ with respect to D at $D^2=-a^2$

Working Rule

If $f(x)=k\sin(ax+b)$, then

$$\text{P.I.} = \frac{\sin(ax+b)}{F(D)} = \frac{\sin(ax+b)}{\phi(D^2) + D\psi(D^2)}$$

$$= \frac{\sin(ax+b)}{\phi(-a^2) + D\psi(-a^2)}$$

$$= \frac{\phi(-a^2) - D\psi(-a^2)}{\left[\phi(-a^2)\right]^2 - D^2\left[\psi(-a^2)\right]^2}\sin(ax+b)$$

$$= \frac{\phi(-a^2) - D\psi(-a^2)}{\left[\phi(-a^2)\right]^2 - (-a^2)\left[\psi(-a^2)\right]^2}\sin(ax+b)$$

$$= \frac{\phi(-a^2)\sin(ax+b) - a\psi(-a^2)\cos(ax+b)}{\left[\phi(-a^2)\right]^2 + a^2\left[\psi(-a^2)\right]^2}, \text{ provided } Dr \neq 0$$

If $Dr = 0$, then

$$\text{P.I.} = x\frac{\sin(ax+b)}{F'(D)\big|_{D^2=-a^2}}, \text{ provided } F'(D)\big|_{D^2=-a^2} \neq 0$$

If $F'(D)\big|_{D^2=-a^2}$, then

$$\text{P.I.} = x^2 \cdot \frac{\sin(ax+b)}{F''(D)\big|_{D^2=-a^2}}, \text{ provided } F''(D)\big|_{D^2=-a^2} \neq 0 \text{ and so on.}$$

Similarly we use the same procedure for $\cos(ax+b)$

Worked Examples

1. *Solve* $y'' - 4y = \cos 2x$

➡ The AE is $m^2 - 4 = 0$

$$m = 2, -2$$

Therefore, $\text{C.F.} = c_1e^{2x} + c_2e^{-2x}$

$$\text{P.I.} = \frac{\cos 2x}{D^2 - 4} = \frac{\cos 2x}{-2^2 - 4} = -\frac{1}{8}\cos 2x$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}\cos 2x$$ ■

2. *Solve* $(D^2 + 5D + 6)y = \sin x$

➡ The AE is $m^2 + 5m + 6 = 0$

$m = -3, -2$

Therefore, C.F. $= c_1 e^{-3x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{\sin x}{D^2 + 5D + 6}$$

$$= \frac{\sin x}{-1^2 + 5D + 6} = \frac{1}{5(D+1)}\sin x$$

$$= \frac{1}{5}\frac{(D-1)}{(D+1)(D-1)}\sin x = \frac{1}{5}\frac{D-1}{(D^2-1)}\sin x$$

$$= \frac{1}{5}\frac{(D-1)}{(-1^2-1)}\sin x = -\frac{1}{10}(\cos x - \sin x)$$

Hence, the general solution is

$$y = c_1 e^{-3x} + c_2 e^{-2x} - \frac{1}{10}(\cos x - \sin x)$$ ■

3. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos 2x$

➡ The A.E is $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

Therefore,

C.F $= c_1 e^{x} + c_2 e^{2x}$

$$\text{P.I} = \frac{\cos 2x}{D^2 - 3D + 2}$$

$$= \frac{\cos 2x}{-2^2 - 3D + 2} = \frac{\cos 2x}{-3D - 2}$$

$$= -\frac{\cos 2x}{3D + 2} \times \frac{3D-2}{3D-2}$$

$$= \frac{-(3D-2)\cos 2x}{9D^2-4}$$

$$= \frac{-(3D-2)\cos 2x}{9(-2^2)-4} = \frac{-(3D-2)\cos 2x}{-40}$$

$$= \frac{1}{40}[3(-\sin 2x \cdot 2) - 2\cos 2x]$$

$$= \frac{1}{40}(-6\sin 2x - 2\cos 2x) = -\frac{1}{20}(3\sin 2x + \cos 2x)$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} - \frac{1}{20}(3\sin 2x + \cos 2x)$$ ■

4. *Solve* $(D^2+1)y = \sin(2x+3)$

➡ The AE is $m^2 + 1 = 0$

$m = \pm i$

Therefore, C.F. $= c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{\sin(2x+3)}{D^2+1} = \frac{\sin(2x+3)}{-2^2+1} = -\frac{1}{3}\sin(2x+3)$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{3}\sin(2x+3)$$ ■

5. *Solve* $(D^2+a^2)y = \cos ax$

➡ The AE is $m^2 + a^2 = 0$

$m = \pm ai$

Therefore, C.F. $= c_1 \cos ax + c_2 \sin ax$

$$\text{P.F.} = \frac{\cos ax}{D^2+a^2} = x \cdot \frac{\cos ax}{2D} = \frac{x}{2}\int \cos ax\,dx$$

$$= \frac{x}{2}\left(\frac{\sin ax}{a}\right) = \frac{x}{2a}\sin ax$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{2a}\sin ax$$ ■

6. *Solve* $\left(D^2 + a^2\right)y = \sin ax$

➡ The AE is $m^2 + a^2 = 0$

$m = \pm ai$

Therefore, $\text{C.F.} = c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{\sin ax}{D^2 + a^2} = x \cdot \frac{\sin ax}{2D} = \frac{x}{2}\int \sin ax\,dx$$

$$= \frac{x}{2}\left(-\frac{\cos ax}{a}\right) = -\frac{x}{2a}\cos ax$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a}\cos ax \quad ■$$

7. *Solve* $\frac{d^2y}{dx^2} + 9y = \sin 3x$

➡ The AE is $m^2 + 9 = 0$

$m = \pm 3i$

Therefore, $\text{C.F.} = c_1 \cos 3x + c_2 \sin 3x$

$$\text{P.I.} = \frac{\sin 3x}{D^2 + 9} = x \cdot \frac{\sin 3x}{2D} = \frac{x}{2}\int \sin 3x\,dx$$

$$= \frac{x}{2}\left(\frac{-\cos 3x}{3}\right) = -\frac{x}{6}\cos 3x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{6}\cos 3x \quad ■$$

8. *Solve* $\left(D^2 + 1\right)y = \cos^2 x$

➡ The AE is $m^2 + 1 = 0$

$m = \pm i$

Therefore, $\text{C.F.} = c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D^2 + 1}\cos^2 x = \frac{1}{D^2 + 1} \cdot \frac{1}{2}\left(1 + \cos 2x\right)$$

$$=\frac{1}{2}\frac{1}{(D^2+1)}+\frac{1}{2}\frac{\cos 2x}{(D^2+1)}=\frac{1}{2}\frac{1}{(0+1)}+\frac{1}{2}\frac{\cos 2x}{(-2^2+1)}$$

$$\text{P.I.}=\frac{1}{2}-\frac{1}{6}\cos 2x$$

Hence, the general solution is

$$y=c_1\cos x+c_2\sin x+\frac{1}{2}-\frac{1}{6}\cos 2x$$ ■

9. *Solve* $\frac{d^2y}{dx^2}-4y=\sin^3 x$

➡ The AE is $m^2-4=0$

$$m=2,-2$$

$$\text{P.I.}=\frac{1}{D^2-4}\sin^3 x=\frac{1}{(D^2-4)}\frac{1}{4}[3\sin x-\sin 3x]$$

$$=\frac{3}{4}\frac{\sin x}{(D^2-4)}-\frac{1}{4}\frac{\sin 3x}{(D^2-4)}=\frac{3}{4}\frac{\sin x}{(-1^2-4)}-\frac{1}{4}\frac{\sin 3x}{(-3^2-4)}$$

$$\text{P.I.}=-\frac{3}{20}\sin x+\frac{1}{52}\sin 3x$$

Hence, the general solution is

$$y=c_1e^{2x}+c_2e^{-2x}-\frac{3}{20}\sin x+\frac{1}{52}\sin 3x$$ ■

10. *Solve* $(D^2-4D+3)y=\sin 3x\cos 2x$

➡ The AE is $m^2-4m+3=0$

$$(m-1)(m-3)=0$$

$$m=1,\ 3$$

Therefore, $\text{C.F.}=c_1e^x+c_2e^{3x}$

$$\text{P.I.}=\frac{1}{D^2-4D+3}\sin 3x\cos 2x$$

$$=\frac{1}{\left(D^2-4D+3\right)}\frac{1}{2}[\sin 5x+\sin x]$$

$$=\frac{1}{2}\frac{\sin 5x}{\left(D^2-4D+3\right)}+\frac{1}{2}\frac{\sin x}{\left(D^2-4D+3\right)}$$

$$=\frac{1}{2}\frac{\sin 5x}{\left(-5^2-4D+3\right)}+\frac{1}{2}\frac{\sin x}{\left(-1^2-4D+3\right)}$$

$$=\frac{1}{2}\frac{1}{(-4D-22)}\sin 5x+\frac{1}{2}\frac{\sin x}{(2-4D)}$$

$$=-\frac{1}{4}\frac{2D-11}{(2D-11)(2D+11)}\sin 5x+\frac{1}{4}\frac{1+2D}{(1-2D)(1+2D)}\sin x$$

$$=-\frac{1}{4}\frac{2D-11}{\left(4D^2-121\right)}\sin 5x+\frac{1}{4}\frac{1+2D}{\left(1-4D^2\right)}\sin x$$

$$=\frac{1}{884}(2D-11)\sin 5x+\frac{1}{20}(1+2D)\sin x$$

$$=\frac{1}{884}[10\cos 5x-11\sin 5x]+\frac{1}{20}[\sin x+2\cos x]$$

Hence, the general solution is

$$y=c_1e^{x}+c_2e^{3x}+\frac{1}{884}[10\cos 5x-11\sin 5x]+\frac{1}{20}[\sin x+2\cos x].$$ ■

11. *Solve* $\dfrac{d^2y}{dx^2}+y=\sin^2 x\cos^2 x$

➡ The AE is $m^2+1=0$

$m=\pm i$

Therefore, C.F. $=c_1\cos x+c_2\sin x$

$$\text{P.I.}=\frac{\sin^2 x\cos^2 x}{D^2+1}=\frac{1}{D^2+1}(\sin x\cos x)^2$$

$$=\frac{1}{D^2+1}\left(\frac{\sin 2x}{2}\right)^2=\frac{1}{4}\frac{1}{\left(D^2+1\right)}\sin^2 2x$$

$$=\frac{1}{4}\frac{1}{\left(D^2+1\right)}\frac{1}{2}(1-\cos 4x)=\frac{1}{8}\frac{1}{\left(D^2+1\right)}-\frac{1}{8}\frac{\cos 4x}{\left(D^2+1\right)}$$

$$=\frac{1}{8}\frac{1}{(0+1)}-\frac{1}{8}\frac{\cos 4x}{(-4^2+1)}=\frac{1}{8}+\frac{1}{120}\cos 4x$$

Hence, the general solution is

$$y=c_1\cos x+c_2\sin x+\frac{1}{8}+\frac{1}{120}\cos 4x$$ ■

12. *Solve* $(D^2-4)y=e^x+\sin 2x$

➡ The AE is $m^2-4=0$

$m=2,-2$

Therefore, $\text{C.F.}=c_1e^{2x}+c_2e^{-2x}$

$$\text{P.I.}=\frac{e^x+\sin 2x}{D^2-4}=\frac{e^x}{D^2-4}+\frac{\sin 2x}{D^2-4}=\frac{e^x}{1^2-4}+\frac{\sin 2x}{-2^2-4}$$

$$\text{P.I.}=-\frac{1}{3}e^x-\frac{1}{8}\sin 2x$$

Hence, the general solution is

$$y=c_1e^{2x}+c_2e^{-2x}-\frac{1}{3}e^x-\frac{1}{8}\sin 2x$$ ■

13. *Solve* $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=e^{2x}-\cos^2 x$

➡ The AE is $m^2+2m+1=0$

$m=1,1$

Therefore, $\text{C.F.}=(c_1+c_2x)e^x$

$$\text{P.I.}=\frac{1}{D^2+2D+1}\left(e^{2x}-\cos^2 x\right)$$

$$=\frac{1}{D^2+2D+1}\left(e^{2x}-\frac{1}{2}(1+\cos 2x)\right)$$

$$=\frac{e^{2x}}{D^2+2D+1}-\frac{1}{2}\frac{1}{(D^2+2D+1)}-\frac{1}{2}\frac{\cos 2x}{(D^2+2D+1)}$$

$$=\frac{1}{9}e^{2x}-\frac{1}{2}\frac{1}{[0+2(0)+1]}-\frac{1}{2}\frac{\cos 2x}{[-2^2+2D+1]}$$

$$=\frac{1}{9}e^{2x}-\frac{1}{2}-\frac{1}{2}\frac{(2D+3)}{(2D-3)(2D+3)}\cos 2x$$

$$=\frac{1}{9}e^{2x}-\frac{1}{2}-\frac{1}{2}\frac{(2D+3)}{(4D^2-9)}\cos 2x$$

$$=\frac{1}{9}e^{2x}-\frac{1}{2}-\frac{1}{2}\frac{(2D+3)}{[4(-2^2)-9]}\cos 2x$$

$$=\frac{1}{9}e^{2x}-\frac{1}{2}+\frac{1}{50}(-4\sin 2x+3\cos 2x)$$

Hence, the general solution is

$$y=(c_1+c_2x)e^{x}+\frac{1}{9}e^{2x}-\frac{1}{2}+\frac{1}{50}(3\cos 2x-4\sin 2x)$$ ■

14. *Solve* $(D^4-2D^2+1)y=\cos x$

➡ The AE is $m^4-2m^2+1=0$

$$(m^2-1)^2=0$$

$$m=1,1,-1,-1$$

Therefore, C.F. $=(c_1+c_2x)e^{x}+(c_3+c_4x)e^{-x}$

$$\text{P.I.}=\frac{\cos x}{D^4-2D^2+1}=\frac{\cos x}{(D^2)^2-2D^2+1}$$

$$=\frac{\cos x}{(-1^2)^2-2(-1^2)+1}=\frac{1}{4}\cos x$$

Hence, the general solution is

$$y=(c_1+c_2x)e^{x}+(c_3+c_4)e^{-x}+\frac{1}{4}\cos x$$ ■