$$\left(a_n D^n + a_{n-1} D^{n-1} + --- + a_2 D^2 + a_1 D + a_0\right) y = f(x) \qquad ---(3)$$

where $D = \frac{d}{dx},\ D^2 = \frac{d^2}{dx^2} --- D^n = \frac{d^n}{dx^n}$

Using the polynomial operator

$F(D) = a_n D^n + a_{n-1} D^{n-1} + --- + a_2 D^2 + a_1 D + a_0$

equation (3) can be written as

$$F(D) y = f(x)$$

If $f(x) = 0$, then the differential equation (1) is called *homogeneous equation* otherwise it is called *non-homogeneous equation*.

Since the general solution of an ordinary differential equation $F(D) y = 0$ of order n contains n orbitrary constants. We conclude that $c_1 y_1 + c_2 y_2 + --- + c_n y_n = u$ is the general solution of $F(D) y = 0$.

Thus $F(D) u = 0$ ---(4)

Again let v be any particular solution of $F(D) y = f(x)$

i.e., $F(D) v = f(x)$ ---(5)

Hence, $F(D)(u + v) = F(D) u + F(D) v$

$= 0 + f(x)$

$$F(D)(u + v) = f(x)$$

This shows that $y = u + v$ is the complete solution of $F(D) y = f(x)$. Here the first part $u = c_1 y_1 + c_2 y_2 + --- + c_n y_n$ called the *complimentary function* and the second part v is called the *particular integral*.

Note It should be remembered that the particular integral appears due to $f(x)$ in $F(D) y = f(x)$. Hence, for non-homogeneous equations the general solution or complete solution is sum of complimentary function and particular integral and for homogeneous equation complementary function will be the general solution.

Auxiliary Equation

Consider the homogeneous linear differential equation of order *n* with constant coefficients

$$F(D)y = 0$$

i.e., $\left(a_n D^n + a_{n-1} D^{n-1} + --- + a_2 D^2 + a_1 D + a_0\right) y = 0$

Let $y = e^{mx}$ be the solution of this equation.

$$\therefore \left(a_n D^n + a_{n-1} D^{n-1} + --- + a_2 D^2 + a_1 D + a_0\right) e^{mx} = 0$$

$$\Rightarrow a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + ---$$

$$+ a_2 m^2 e^{mx} + a_1 m e^{mx} + a_0 e^{mx} = 0$$

$$\Rightarrow \left(a_n m^n + a_{n-1} m^{n-1} + --- + a_2 m^2 + a_1 m + a_0\right) e^{mx} = 0$$

$$\Rightarrow a_n m^n + a_{n-1} m^{n-1} + --- + a_2 m^2 + a_1 m + a_0 = 0 \quad \text{---(6)}$$

This is called the auxiliary equation.

Complimentary Function (C.F.)

Let equation (6) be the auxiliary equation of the differential equation (3) and roots of equation (6) be $m_1, m_2, m_3, -----, m_n$

Case (i) If the roots of the equation (6) are real and distinct,

i.e., $m_1 \neq m_2 \neq m_3 \neq --- \neq m_n$,

then the complimentary function of the differential equation (3) is,

C.F. $= c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + ---- + c_n e^{m_n x}$

Case (ii) If the roots of the equation (6) are real and repeated

i.e., $m_1 = m_2 = m_3 = --- = m_n$,

then the complimentary function of the differential equation (3) is,

C.F. $= \left(c_1 + c_2 x + c_3 x^2 + ------ + c_n x^{n-1}\right) e^{m_1 x}$

Case (iii) If the roots of the equation (6) are complex, i.e.,

$m_1, m_2 = \alpha \pm i\beta$

then the complimentary function of the differential equation (3) is,

C.F. $= e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$

[If the equation (3) is 2nd order differential equation]

Case (iv) If the roots of the equation (6) are complex and non repeated i.e., $m_1, m_2 = \alpha_1 \pm i\beta_1$ and $m_3, m_4 = \alpha_2 \pm i\beta_2$,
then the complimentary function of equation (3) is,
C.F. $= e^{\alpha_1 x}\left[c_1 \cos\beta_1 x + c_2 \sin\beta_1 x\right] + e^{\alpha_2 x}\left[c_3 \cos\beta_2 x + c_4 \sin\beta_2 x\right]$
[If the equation (3) is 4th order differential equation]

Case (v) If the roots of the equation (6) are complex and repeated, i.e., $m_1, m_2 = \alpha \pm i\beta = m_3, m_4$
then the complimentary function of equation (3) is,
C.F. $= e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right]$

Worked Examples

1. *Solve* $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0$

➡ The given equation can be written as

$$(D^2 - 2D - 3)\,y = 0$$

The A.E is $m^2 - 2m - 3 = 0$

$$m^2 - 3m + m - 3 = 0$$
$$m(m-3) + 1(m-3) = 0$$
$$(m+1)(m-3) = 0 \therefore m = -1 \text{ or } m = 3$$

Here, the roots are real and distinct
Therefore,

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{3x}$$

Hence, the general solution of the given differential equation is

$$y = c_1 e^{-x} + c_2 e^{3x}$$ ■

2. *Solve* $y'' + y' + y = 0$

➡
$$y'' + y' + y = 0$$
$$\left(D^2 + D + 1\right)y = 0$$

The A.E is $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

Therefore,

$$\text{C.F.} = e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

Hence, $y = e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$ ■

3. *Solve* $(D^2 - 2D + 1)y = 0$

➡ $(D^2 - 2D + 1)y = 0$

The A.E is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$\therefore m = 1, 1$$

Here the roots are real and repeated

Therefore, $\text{C.F.} = (c_1 + c_2 x)e^x$

Hence, $y = (c_1 + c_2 x)e^x$ ■

4. *Solve* $(D^2 + 5D + 1)y = 0$

➡ The AE is $m^2 + 5m + 1 = 0$

$$m = \frac{-5 \pm \sqrt{25 - 4(1)(1)}}{2(1)} = \frac{-5 \pm \sqrt{21}}{2}$$

$$m = \frac{-5 + \sqrt{21}}{2}, \frac{-5 - \sqrt{21}}{2}$$

Therefore,

$$\text{C.F.} = c_1 e^{\left(\frac{-5+\sqrt{21}}{2}\right)x} + c_2 e^{\left(\frac{-5-\sqrt{21}}{2}\right)x}$$

Hence, $y = c_1 e^{\left(\frac{-5+\sqrt{21}}{2}\right)x} + c_2 e^{\left(\frac{-5-\sqrt{21}}{2}\right)x}$ ■

5. *Solve* $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=0$ *given that* $y=0$ *and* $\frac{dy}{dx}=6$ *when* $x=0$

➡ The given equation can be written as

$$\left(D^2-4D-5\right)y=0$$

The AE is $m^2-4m-5=0$

$$m^2-5m+m-5=0$$
$$m(m-5)+1(m-5)=0$$
$$(m+1)(m-5)=0$$
$$m=-1,\ 5$$

Therefore, $\text{C.F.}=c_1e^{-x}+c_2e^{5x}$

Thus, $y=c_1e^{-x}+c_2e^{5x}$ ---(1)

Put $y=0$ and $x=0$

$0=c_1+c_2$ ---(2)

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx}=-c_1e^{-x}+5c_2e^{5x}$$

Put $x=0$ and $\frac{dy}{dx}=6$

$6=-c_1+5c_2$ ---(3)

$$(2)+(3)\Rightarrow 6=6c_2\Rightarrow c_2=1$$
$$\therefore c_1=-1$$
$$y=-e^{-x}+e^{5x}$$

■

6. *Solve* $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$, *given that* $x(0)=0,\ \frac{dx}{dt}(0)=1$

➡ $\left(D^2-3D+2\right)x=0$

The A.E is $m^2-3m+2=0$

$$m=1,2$$

Therefore,

$$\text{C.F.}=c_1e^{t}+c_2e^{2t}$$
$$x=c_1e^{t}+c_2e^{2t}$$

$$x(0) = c_1(1) + c_2(1)$$

$$0 = c_1 + c_2 \Rightarrow c_1 = -c_2 \qquad \text{---(1)}$$

$$\frac{dx}{dt} = c_1 e^t + 2c_2 e^{2t}$$

$$\frac{dx(0)}{dt} = c_1 + 2c_2$$

$$1 = -c_2 + 2c_2 \Rightarrow c_2 = 1 \Rightarrow c_1 = -1$$

Hence, $x = -e^t + e^{2t}$ ■

7. *Solve* $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

➡ $(D^3 - 6D^2 + 11D - 6)y = 0$

The A.E is $m^3 - 6m^2 + 11m - 6 = 0$

Here, $m = 1$ is one of the roots

$m=1$	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

Therefore,

$$m^2 - 5m + 6 = 0$$

$$m^2 - 2m - 3m + 6 = 0$$

$$m(m-2) - 3(m-2) = 0$$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

$$m = 1, 2, 3$$

Hence,

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$ ■

8. *Solve* $(D^3 - 4D^2 + 5D - 2)y = 0$

➡ $(D^3 - 4D^2 + 5D - 2)y = 0$

The A.E is

$$m^3 - 4m^2 + 5m - 2 = 0$$

by inspection method 1 is one of the roots.

$m = 1$	1	-4	5	-2
	0	1	-3	2
	1	-3	2	0

$$1.m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1,1,2$$

Therefore,

$$\text{C.F.} = (c_1 + c_2 x)\, e^x + c_3 e^{2x}$$

Hence, $y = (c_1 + c_2 x)\, e^x + c_3 e^{2x}$ ■

9. *Solve* $\dfrac{d^3 y}{dx^3} + y = 0$

➡ The given equation can be written as

$$(D^3 + 1)y = 0$$

The AE is $m^3 + 1 = 0$

$$(m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Therefore, $\text{C.F.} = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

Thus,

$$y = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$ ■

10. *Solve* $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} - 2y = 0$

➡ The given equation can be written as

$$\left(D^3 - 3D - 2\right)y = 0$$

The AE is $m^3 - 3m - 2 = 0$

Now $m = -1$ is one of the roots

$m=-1$	1	0	-3	-2
	0	-1	1	2
	1	-1	-2	0

Therefore, $m^2 - m - 2 = 0$

$$m^2 - 2m + m - 2 = 0$$
$$m(m-2) + 1(m-2) = 0$$
$$(m+1)(m-2) = 0$$
$$m = -1, 2$$
$$\therefore m = -1, -1, 2$$

Therefore,

$$\text{C.F.} = (c_1 + c_2x)e^{-x} + c_3e^{2x}$$

Thus, $y = (c_1 + c_2x)e^{-x} + c_3e^{2x}$ ■

11. *Solve* $\left(D^4 - 2D^3 + 5D^2 - 8D + 4\right)y = 0$

➡ $\left(D^4 - 2D^3 + 5D^2 - 8D + 4\right)y = 0$

The A.E is

$$m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$$

$m = 1$ is one of the roots

$m=1$	1	-2	5	-8	4
	0	1	-1	4	-4
	1	-1	4	-4	0

$$\therefore m^3 - m^2 + 4m - 4 = 0$$

$$m^2(m-1)+4(m-1)=0$$
$$(m^2+4)(m-1)=0$$
$$m=\pm 2i,\ 1$$

$\therefore m=1,\ 1,\ \pm 2i$

Therefore,

$$\text{C.F.}=(c_1+c_2x)e^x+c_3\cos 2x+c_4\sin 2x$$

Hence, $\quad y=(c_1+c_2x)e^x+c_3\cos 2x+c_4\sin 2x$ ■

12. *Solve* $(D^4+4D^3-5D^2-36D-36)y=0$

➡ The AE is $m^4+4m^3-5m^2-36m-36=0$

Here $m=-2$ is one of the roots of the above equation.

$m=-2$	1	4	-5	-36	-36
	0	-2	-4	18	36
$m=-2$	1	2	-9	-18	0
	0	-2	0	18	
	1	0	-9	0	

$$m^2-9=0$$
$$m=\pm 3$$
$$\therefore m=-2,\ -2,\ -3,\ 3$$

Therefore, C.F $=(c_1+c_2x)e^{-2x}+c_3e^{-3x}+c_4e^{3x}$

Hence, $y=(c_1+c_2x)e^{-2x}+c_3e^{-3x}+c_4e^{3x}$ ■

13. *If* $\dfrac{d^4y}{dx^4}-m^4y=0$, *then prove that*

$$y=k_1\sinh mx+k_2\cosh mx+k_3\cos mx+k_4\sin mx$$

➡ The given equation can be written as

$$(D^4-m^4)y=0$$

The AE is $M^4-m^4=0$

$$(M^2+m^2)(M^2-m^2)=0 \Rightarrow M=\pm m,\ \pm mi$$

Therefore, C.F. $=c_1e^{mx}+c_2e^{-mx}+c_3\cos mx+c_4\sin mx$

We have, $\sinh mx = \dfrac{e^{mx} - e^{-mx}}{2}$

$$\cosh mx = \frac{e^{mx} + e^{-mx}}{2}$$

$$\Rightarrow e^{mx} = \sinh mx + \cosh mx$$

$$e^{-mx} = \cosh mx - \sinh mx$$

Hence,

$$y = c_1(\sinh mx + \cosh mx) + c_2(\cosh mx - \sinh mx) + c_3 \cos mx + c_4 \sin mx$$

$$= (c_1 - c_2)\sinh mx + (c_1 + c_2)\cosh mx + c_3 \cos mx + c_4 \sin mx$$

$$y = k_1 \sinh mx + k_2 \cosh mx + k_3 \cos mx + k_4 \sin mx$$ ■

14. *Solve* $\dfrac{d^4y}{dx^4} + a^4 y = 0$

➡ The given equation can be written as

$$(D^4 + a^4)y = 0$$

The AE is $m^4 + a^4 = 0$

$$m^4 + a^4 + 2m^2a^2 - 2m^2a^2 = 0$$

$$(m^2 + a^2)^2 - (\sqrt{2}ma)^2 = 0$$

$$(m^2 + a^2 - \sqrt{2}ma)(m^2 + a^2 + \sqrt{2}ma) = 0$$

$$\Rightarrow m^2 - \sqrt{2}ma + a^2 = 0 \text{ or } m^2 + \sqrt{2}ma + a^2 = 0$$

$$\Rightarrow m = \frac{\sqrt{2}a \pm \sqrt{2a^2 - 4a^2}}{2(1)} \qquad m = \frac{-\sqrt{2}a \pm \sqrt{2a^2 - 4a^2}}{2(1)}$$

$$= \frac{\sqrt{2}a \pm \sqrt{2}ai}{2} \qquad = \frac{-\sqrt{2}a \pm \sqrt{2}ai}{2}$$

$$= \frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i \qquad = -\frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i$$

$$\therefore m = \frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i, \quad -\frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i$$

Therefore,

$$\text{C.F.} = e^{\frac{a}{\sqrt{2}}x}\left[c_1\cos\left(\frac{a}{\sqrt{2}}x\right)+c_2\sin\left(\frac{a}{\sqrt{2}}x\right)\right]$$

$$+e^{-\frac{a}{\sqrt{2}}x}\left[c_3\cos\left(\frac{a}{\sqrt{2}}x\right)+c_4\sin\left(\frac{a}{\sqrt{2}}x\right)\right]$$

Hence,

$$y = e^{\frac{a}{\sqrt{2}}x}\left[c_1\cos\left(\frac{a}{\sqrt{2}}x\right)+c_2\sin\left(\frac{a}{\sqrt{2}}x\right)\right]$$

$$+e^{-\frac{a}{\sqrt{2}}x}\left[c_3\cos\left(\frac{a}{\sqrt{2}}x\right)+c_4\sin\left(\frac{a}{\sqrt{2}}x\right)\right] \blacksquare$$

Exercises

Solve the following differential equations

1) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}-3y=0$ 2) $y''-6y'+13y=0$

3) $y''+10y'+25y=0$

4) $(D^2+8)y=0$, *given that* $y(0)=1$ *and* $y'(0)=2\sqrt{2}$

5) $y''+y'-2y=0$, *given that* $y=0$ *and* $y'=3$ for $x=0$

6) $y''+y=0$ *given that* $y(0)=2,\ y\left(\dfrac{\pi}{2}\right)=-2$

7) $y''-3y'+2y=0$ *if* $y(0)=-1,\ y'(0)=0$

8) $8\dfrac{d^2y}{dx^2}-10\dfrac{dy}{dx}+3y=0$ *if* $y(0)=-1,\ y'(0)=0$

9) $y''+4y'+4y=0$ *given that* $y=0,\ y'=-1$ *at* $x=1$

10) $\dfrac{d^2y}{dx^2}-2\sec\alpha\dfrac{dy}{dx}+y=0$ 11) $y''-2\cos\alpha y'+y=0$

12) $\dfrac{d^2y}{dx^2}+2a\dfrac{dy}{dx}+by=0,\ a^2<b$

13) $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-8y=0$ 14) $\dfrac{d^3y}{dx^3}-y=0$

15) $\left(D^3 + D^2 - 7D - 15\right)y = 0$

16) $y''' - 2y'' - y' + 2y = 0$

17) $y''' - 2y'' - a^2 y' + 2a^2 y = 0$

18) $y''' - 3ay'' + 3a^2 y' - a^3 y = 0$

19) $y''' - 3y'' + 4y' - 2y = 0$

20) $\left(D^4 + 2D^3 - 2D - 1\right)y = 0$

21) $\left(D^4 - 2D^3 + 2D^2 - 2D + 1\right)y = 0$

22) $\left(D^4 + 2D^3 - 3D^2 - 4D + 4\right)y = 0$

23) $\left(D^4 - 3D^3 + D^2 - D\right)y = 0$

24) $\left(D^4 - 4D^3 + 6D^2 - 4D + 1\right)y = 0$

25) $\left(D^4 - 12D^2 + 27\right)y = 0$

26) $\left(D^4 + 2a^2 D^2 + a^4\right)y = 0$

27) $\left(D^4 + 3D^2 - 4\right)y = 0$

28) $\left(D^4 + 2D^2 + 1\right)y = 0$

29) $\left(D^4 - D^3 - 3D^2 + 5D - 2\right)y = 0$

30) $\left(D^4 - 6D^3 + 13D^2 - 12D + 4\right)y = 0$

31) $\left(D^4 + 8D^2 + 16\right)y = 0$

32) $\dfrac{d^4y}{dx^4} - 5\dfrac{d^2y}{dx^2} + 4y = 0$

33) $\dfrac{d^4y}{dx^4} - 64y = 0$

Answers

1) $y = c_1 e^{3x} + c_2 e^{-x}$

2) $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

3) $y = (c_1 + c_2 x)e^{5x}$

4) $y = \cos 2\sqrt{2}x + \sin 2\sqrt{2}x$

5) $y = e^x - e^{-2x}$

6) $y = 2(\cos x - \sin x)$

7) $y = e^{2x} - 2e^x$

8) $y = 3e^{\frac{x}{2}} - 3e^{\frac{3x}{4}}$

9) $y = (1 - x)e^{2(1-x)}$

10) $y = c_1 e^{(\sec\alpha + \tan\alpha)x} + c_2 e^{(\sec\alpha - \tan\alpha)x}$

11) $y = e^{x\cos\alpha}\left[c_1 \cos(x\sin\alpha) + c_2 \sin(x\sin\alpha)\right]$

12) $y = e^{-ax}\left[c_1 \cos\left(\sqrt{a^2 - bx}\right) + c_2 \sin\left(\sqrt{a^2 - bx}\right)\right]$

13) $y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$

14) $y = c_1 e^{x} + e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right]$

15) $y = c_1 e^{3x} + e^{-2x}\left(c_2 \cos x + c_3 \sin x\right)$

16) $y = c_1 e^{2x} + c_2 e^{x} + c_3 e^{-x}$

17) $y = c_1 e^{2x} + c_2 e^{ax} + c_3 e^{-ax}$

18) $y = \left(c_1 + c_2 + c_3 x^2\right) e^{ax}$

19) $y = c_1 e^{x} + e^{x}\left(c_2 \cos x + c_3 \sin x\right)$

20) $y = c_1 e^{x} + \left(c_1 + c_2 x + c_3 x^2\right) e^{-x}$

21) $y = c_1 \cos x + c_2 \sin x + \left(c_3 + c_4 x\right) e^{x}$

22) $y = (c_1 + c_2 x) e^{x} + (c_3 + c_4 x) e^{-2x}$

23) $y = c_1 + \left(c_2 + c_3 x + c_4 x^2\right) e^{x}$

24) $y = \left(c_1 + c_2 x + c_3 x^2 + c_4 x^3\right) e^{x}$

25) $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 e^{\sqrt{3}x} + c_4 e^{-\sqrt{3}x}$

26) $y = (c_1 + c_2 x)\cos ax + (c_3 + c_4 x)\sin ax$

27) $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$

28) $y = \left(c_1 + c_2 x\right)\cos x + \left(c_3 + c_4 x\right)\sin x$

29) $y = \left(c_1 + c_2 x + c_3 x^2\right) e^{x} + c_4 e^{-2x}$

30) $y = \left(c_1 + c_2 x\right) e^{x} + \left(c_3 + c_4 x\right) e^{2x}$

31) $y = \left(c_1 + c_2 x\right)\cos 2x + \left(c_3 + c_4 x\right)\sin 2x$

32) $y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$

33) $y = c_1 e^{2\sqrt{2}x} + c_2 e^{-2\sqrt{2}x} + c_3 \cos 2\sqrt{2}x + c_4 \sin 2\sqrt{2}x$