8. *Solve* $(x-y)^2\left(1+\frac{dy}{dx}\right)=(x+y)^2\left(1-\frac{dy}{dx}\right)$

➡ $(x-y)^2\left(1+\frac{dy}{dx}\right)=(x+y)^2\left(1-\frac{dy}{dx}\right)$

Put $x-y=t$ and $x+y=z$

$$1-\frac{dy}{dx}=\frac{dt}{dx} \qquad 1+\frac{dy}{dx}=\frac{dz}{dx}$$

Substituting the above equations in the given equation, we get

$$t^2\frac{dz}{dx}=z^2\frac{dt}{dx}$$

$$\frac{dt}{t^2}=\frac{dz}{z^2}$$

Integrating the above equation, we get

$$\int\frac{dt}{t^2}=\int\frac{dz}{z^2}+c$$

$$\int t^{-2}dt=\int z^{-2}dz+c$$

$$\frac{t^{-1}}{-1}=\frac{z^{-1}}{-1}+c$$

$$-\frac{1}{t}+\frac{1}{z}=c$$

$$-\frac{1}{(x-y)}+\frac{1}{(x+y)}=c$$ ■

9. *Solve* $x^2y^2(xdy+ydx)=e^{xy}\sec^2(x+y)(dx+dy)$

➡ $x^2y^2(xdy+ydx)=e^{xy}\sec^2(x+y)(dx+dy)$

$$(xy)^2d(xy)=e^{xy}\sec^2(x+y)d(x+y)$$

Put $xy=t$ $x+y=z$

$$t^2dt=e^t\sec^2 zdz$$

$$\int\frac{t^2}{e^t}dt=\int\sec^2 zdz+c$$

$$\int t^2e^{-t}dt=\tan z+c$$

$$t^2\frac{e^{-t}}{-1}-\int 2t\frac{e^{-t}}{-1}dt=\tan(x+y)+c$$

$$-t^2e^{-t}+2t\frac{e^{-t}}{-1}-\int 2\frac{e^{-t}}{-1}dt=\tan(x+y)+c$$

$$-t^2e^{-t}-2te^{-t}+2\frac{e^{-t}}{-1}=\tan(x+y)+c$$

$$-e^{-t}\left(t^2+2t+2\right)=\tan(x+y)$$

$$e^{-t}\left(t^2+2t+2\right)+\tan(x+y)=c_1$$

$$e^{-xy}\left(x^2y^2+2xy+2\right)+\tan(x+y)=c_1$$ ■

10. *Solve* $(x+y)^2(dx+dy)=\sec xy\tan xy(xdy+ydx)$

➡ $(x+y)^2(dx+dy)=\sec xy\tan xy(xdy+ydx)$

$\Rightarrow (x+y)^2 d(x+y)=\sec xy\tan xy\, d(xy)$

Put $x+y=t$ $\quad xy=z$

$t^2dt=\sec z\tan zdz$

$\int t^2dt=\int \sec z\tan zdz+c$

$\frac{t^3}{3}=\sec z+c$

$\frac{1}{3}(x+y)^3=\sec xy+c$ ■

11. *Solve*

$[\cos x\tan y+\cos(x+y)]dx+\left[\sin x\sec^2 y+\cos(x+y)\right]dy=0$

➡ $[\cos x\tan y+\cos(x+y)]dx+\left[\sin x\sec^2 y+\cos(x+y)\right]dy=0$

$\cos x\tan ydx+\cos(x+y)dx+\sin x\sec^2 ydy+\cos(x+y)dy=0$

$\tan y\cos xdx+\sin x\sec^2 ydy+\cos(x+y)d(x+y)=0$

$\tan yd(\sin x)+\sin xd(\tan y)+\cos(x+y)d(x+y)=0$

$\Rightarrow d(\tan y\sin x)+\cos tdt=0$, where $t=x+y$

$\Rightarrow \int d(\tan y\sin x)+\int\cos tdt=c$

$\tan y\sin x+\sin t=c$

i.e., $\tan y\sin x+\sin(x+y)=c$ ■

12. *Solve* $xdx + ydy + \dfrac{xdy - ydx}{x^2 + y^2} = 0$

➡ $xdx + ydy + \dfrac{xdy - ydx}{x^2 + y^2} = 0$

$$xdx + ydy + \frac{xdy - ydx}{x^2\left[1 + \dfrac{y^2}{x^2}\right]} = 0$$

$$xdx + ydy + \frac{d(y/x)}{1 + (y/x)^2} = 0 \quad \text{where } d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$\int xdx + \int ydy + \int \frac{dt}{1+t^2} = c$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1} t = c$$

$$\frac{x^2 + y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c$$

■

Exercises

1) $(x^2 + y^2 + x)dx + xdy = 0$
2) $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$
3) $x^2ydx - (x^3 + y^3)dy = 0$
4) $(x^4 + y^4)dx - xy^3dy = 0$
5) $y(x + y)dx - x^2dy = 0$
6) $(x^3 - 3xy^2)dx - (y^3 - 3x^2y)dy = 0$
7) $(8ydx + 8xdy) + x^2y^3(4ydx + 5xdy) = 0$
8) $(5x^3 + 3xy + 2y^2)dx + (x^2 + 2xy)dy = 0$
9) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$
10) $(xy\sin xy + \cos xy)\,ydx + (xy\sin xy - \cos xy)\,xdy = 0$
11) $ydx - xdy + \log x\,dx = 0$
12) $(1 + xy)\,ydx + (1 - xy)\,xdy = 0$

13) $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$

14) $\left(y-3x^2y^2e^{x^3}\right)dx-xdy=0$

15) $ydx+x\log xdy=0$

Answers

1) $\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^2y^2}{2}=c$

2) $x^2e^y+\left(\dfrac{x^2}{y}\right)+\left(\dfrac{x}{y^3}\right)=c$

3) $\sin^{-1}(y/x)=\log x+c$

4) $y^4=4x^4\log x+cx^4$

5) $\dfrac{x}{y}+\log x=c$

6) $\left(x^2+y^2\right)^4=c\left(x^2-y^2\right)^2$

7) $4x^2y^2+x^4y^5=c$

8) $x^2y^2\left(y^2-x^2\right)=c$

9) $\left(y+\dfrac{2}{y^2}\right)x+y^2=c$

10) $x\sec xy=cy$

11) $y+cx+\log x+1=0$

12) $\log\left(\dfrac{x}{y}\right)-\left(\dfrac{1}{xy}\right)=c$

13) $\left(\dfrac{1}{xy}\right)=c+\log\left(\dfrac{x^2}{y}\right)$

14) $(x/y)-e^{x^3}=c$

15) $y=\dfrac{c}{\log x}$

Higher Order Linear Differential Equations

A differential equation of the form

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+---+a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=f(x) \qquad \text{---(1)}$$

where $a_n, a_{n-1} ---, a_2, a_1, a_0$ and $f(x)$ are constants or functions of x only. If $a_n, a_{n-1} ---, a_2, a_1, a_0$ are constants, then equation (1) is called *linear differential equation with constant coefficients*. The equation (1) can also be written as

$$a_nD^ny+a_{n-1}D^{n-1}y+---+a_2D^2y+a_1Dy+a_0y=f(x) \qquad \text{---(2)}$$